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FAKULTÄT FÜR MATHEMATIK
Dekan Univ.-Prof. Dr. Radu Ioan Boţ

Einladung zur öffentlichen Defensio

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Thema der Dissertation

Functional analysis in spaces of generalized functions

Abstract:
The primary focus of this thesis is to develop and generalize fundamental results in functional analysis within the context of spaces of generalized functions. The central framework employed is that of modules over the ring of Robinson-Colombeau generalized numbers, encompassing Colombeau spaces of generalized functions and spaces of generalized smooth functions. The latter spaces extend Colombeau algebras, sharing many nonlinear properties with ordinary smooth functions, such as closure under composition, a good integration theory, and several classical calculus theorems. In addition, an independent secondary aim, not related to Colombeau-like approach, is the study of weighted Sobolev spaces and their applications to degenerate elliptic partial differential equations, a topic that has been developed independently by the author, focusing on addressing a new approach to weighted Sobolev spaces. This new approach is particularly useful when the weight functions are arbitrarily small. It replaces the conventional approach of excluding points where weight functions are very small by introducing a novel notion of weak derivative. In this new setting, non-locally integrable functions are considered in these spaces. Indeed, we provide conditions under which a degenerate elliptic partial differential equation has a unique non-locally integrable solution. Tools like a Poincaré inequality and a trace operator are developed, and density results of smooth functions are established.
In the first part related to the framework of generalized functions, we establish a suitable version of the Hahn-Banach theorem within the framework of Colombeau spaces. We address the case where the maps are defined by a net of representatives, which simplifies the framework and makes the extension of linear functionals more manageable. As an application of our primary result, we generalize the separation of convex sets property within Colombeau spaces.
In the second part, we explore universal properties as fundamental tools for solving mathematical problems. By means of several examples, we show that universal properties

provide the simplest solutions to various mathematical problems. Specifically, we present the co-universal property of Schwartz distributions as the simplest method for obtaining derivatives of continuous functions, Colombeau algebra as the simplest quotient algebra where representatives of zero are infinitesimal, and generalized smooth functions as the universal framework for associating set-theoretical maps of non-Archimedean numbers defined by nets of smooth functions, such as regularizations of distributions, and possessing arbitrary derivatives. Each of these properties leads to a characterization up to isomorphisms of the corresponding space.
In the third part, we proved the Banach fixed point theorem in spaces of generalized functions, focusing on infinite iterations version that involves hyperfinite sequences and two gauges instead of sequences, since the finite iterations version is very limited. A key application of this extended theorem is the Picard-Lindelöf theorem in spaces of generalized continuous functions. These results extend Banach fixed point theory to new contexts and open avenues for further research in the analysis of generalized function spaces.
In the fourth part, we establish an extension property for uniformly continuous generalized smooth functions. Specifically, this result proves that a generalized smooth function which is uniformly continuous together with all its derivatives on a strongly internal set, can be uniquely extended to the boundary as a generalized smooth function. This result can be used to show that maximal solutions of differential equations have at least one derivative going to infinity at a boundary point.

Prüfungssenat

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Montag, 17. Februar 2025, 14:00 Uhr

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