The Sum-Product Phenomenon via Continuous Real Dimension

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1. Introduction and Historical Context

The sum-product phenomenon is a core theme in additive combinatorics, asserting that a subset A of a ring or field cannot be simultaneously close to being an additive subgroup and a multiplicative subgroup. In other words, both the sumset $A + A$ and the productset AA cannot remain small unless A itself structurally mirrors an existing algebraic subfield.

This behavior was first classically formalized for finite sets of integers, establishing that there exist constants $C, \delta > 0$ such that:

$$\max\{|A + A|, |AA|\} \geq C|A|^{1+\delta} \tag{1}$$

In fields of finite characteristic, the landscape is inherently more intricate due to the presence of actual subfields. Classical extensions of this phenomenon to finite fields demonstrate that if a subset A does not structurally mimic a subfield, it must exhibit significant expansion under either addition or multiplication.

1.1. The Model-Theoretic Approach

This talk introduces a model theoretic generalization of the sum-product phenomenon. Instead of quantifying the "size" of a set using traditional counting measures or discrete cardinalities, we transition to a model-theoretic framework where size is governed by a **continuous real dimension** encoded within first-order logic. We demonstrate that small expansion under addition and multiplication implies the presence of an explicitly **definable subfield** of comparable dimension.

2. Axiomatizing Continuous Real Dimension

Let T be a complete first-order theory and $M \models T$ a model. A *continuous real dimension* is a mapping $\delta : Def(M) \to \mathbb{R}_{\geq 0} \cup \{+\infty\}$ on the collection of parameter-definable sets satisfying the following axioms for any $X, Y \in Def(M)$:

- **Union:** $\delta(X \cup Y) = \max\{\delta(X), \delta(Y)\}$
- **Product:** $\delta(X \times Y) = \delta(X) + \delta(Y)$
- **Finiteness:** $\delta(X) = 0$ if X is a finite set
- **Intersection:** $\delta(X \cap Y) \leq \min\{\delta(X), \delta(Y)\}$
- **Invariance:** $\delta(X) = \delta(Y)$ if there exists a definable bijection $f : X \to Y$

- **Subadditivity** For any definable surjection $f : X \to Y$,

$$\delta(X) = \sup\{\alpha + \beta : \alpha \in \mathbb{R} \cup \{\infty\}, \beta \leq \{z \in Y : \delta(f^{-1}(z)) \geq \alpha\}\}.$$

- **Continuity:** for any formula $\phi(\bar{x}, \bar{y})$ and $s < t \in \mathbb{R}$, there is a $\emptyset-$definable D such that

$$\{a : \delta(\phi(\bar{x}, a)) \leq s\} \subseteq D \subseteq \{b : \delta(\phi(\bar{x}, b)) < t\}.$$

This dimension generalizes classical geometric dimensions; for instance, o-minimal and geometric theories using algebraic closure (acl) dimension, alongside some theories of finite Morley rank. Importunately, it captures coarse pseudo-finite dimension, which treats asymptotic bounds of finite sets within an ultraproduct as a real-valued dimension.

Using this framework, we extend δ to partial types by taking the infimum over defining formulas, which allows us to naturally recover robust model-theoretic definitions for **genericity** and **independence** ($\vec{a} \perp_A \vec{b}$).

3. Combinatorial Machinery in the Dimensional Setting

We adopt some tools of classical additive combinatorics into continuous dimensions.

3.1. Generalized Ruzsa Triangle Inequality

We prove that for type-definable sets X, Y, Z of finite dimension in a dimensional group, the following holds:

$$\delta(X) + \delta(Y - Z) \leq \delta(X - Y) + \delta(X - Z) \tag{2}$$

The proof relies on generic independent realizations, the underlying symmetry and additivity of the dimension function.

3.2. Plünnecke-Ruzsa Sumset Estimates

We adapt Plünnecke's original strategy of utilizing commutative layered graphs. By tracking injections within a definably commutative 3-layered graph $\Gamma = (U_0 \sqcup U_1 \sqcup U_2, E)$, we establish that if a type-definable set expands minimally at the first layer, its higher expansions are bounded. This yields the Sumset Estimate: if $\delta(X + Y) = \delta(X) < \infty$, then for any $m, n \in \mathbb{N}$, $\delta(mY - nY) \leq \delta(X)$.

3.3. Balog-Szemerédi-Gowers (BSG) Lemma & The Katz-Tao Theorem

Using n-gons and triangles we proof a weak variant of the BSG lemma. This serves as the bridge to proving the dimensional version of the **Katz-Tao Lemma**. We show that if a type-definable set X has identical sum and product dimensions ($0 < \delta(X) = \delta(X + X) = \delta(XX) < \infty$), then for any generic types p, q on X, the difference of their products expands minimally:

$$\delta(pq - pq) = \delta(X) \tag{3}$$

4. The Main Theorem: Constructing the Definable Field

Theorem 1. *Let F be a sufficiently saturated model of a theory expanding the theory of fields, equipped with a continuous real dimension δ. Let X be a complete type over a small parameter set A satisfying $0 < \delta(X + X) = \delta(XX) = \delta(X) < \infty$. Let p be a generic type on X over A, and $Y = a^{-1}p$ for some $a \models p$.*

Then $\frac{Y-Y}{Y-Y}$ is a ***definable field****, and $\delta\left(\frac{Y-Y}{Y-Y}\right) = \delta(X)$.*

4.1. Outline of Proof Strategy

(1) **The $(\star)$ Property:** We define a stabilizing property for any element $x \in F$: x satisfies $(\star)$ if $\delta(xY + Y) = \delta(Y)$. Using the dimensonal Ruzsa triangle inequality, we prove that the collection of elements satisfying $(\star)$ forms a field.

(2) **Equivalence to the Quotient Set:** We show via a non-injectivity argument that an element x satisfies $(\star)$ if and only if it can be written as a quotient of differences: $x \in \frac{Y-Y}{Y-Y}$.

(3) **Definability via Subfield Stabilization Theorems:** While the quotient set is initially type-definable, model-theoretic structural properties of type-definable subfields of finite dimension inside a dimensional field ensure that it is automatically first-order definable over the same parameters.

5. Finitary Applications and Conclusion

The advantage of this model-theoretic approach is its generalizability across arbitrary fields. By applying our main theorem to coarse pseudo-finite dimension and invoking ultraproduct structures, we immediately recover a clean, general finitary statement across all fields.

Corollary 1. *For any $\delta \in (0, 1)$, there is $\epsilon(\delta) \in (0, 1)$ and $N(\delta) \in \mathbb{N}$ such that for any finite subset $A \subseteq F$ of a field, if $|A + A|, |AA| \leq |A|^{1+\epsilon(\delta)}$ and $|A| \geq N(\delta)$, then there is a subfield $E \subseteq F$ satisfying:*

$$|A|^{1-\delta} \leq |E| \leq |A|^{1+\delta} \tag{4}$$