Structurable equivalence relations, Borel combinatorics, and countable model theory

Ronnie Chen (University of Michigan)

joint with Alexander Kechris (California Institute of Technology)
and Rishi Banerjee (University of Illinois Chicago)

Universität Wien, November 28, 2024

Locally countable Borel combinatorics

. . . is the study of Borel combinatorial structures (e.g., trees, group actions) on standard Borel spaces (e.g., $\mathbb{R}, 2^{\mathbb{N}}$) which "live on" countable pieces.

Locally countable Borel combinatorics

…is the study of Borel combinatorial structures (e.g., trees, group actions) on standard Borel spaces (e.g., $\mathbb{R}, 2^{\mathbb{N}}$) which "live on" countable pieces.

Example A Borel action $\mathbb{F}_2 = \langle a, b \rangle \curvearrowright X$ induces a Schreier graph on X:

$$x \; G \; y \;\; :\Longleftrightarrow \;\; (ax = y) \vee (bx = y) \vee (ay = x) \vee (by = x)$$

If the action is free, then the graph is a forest (each component is a tree).

Locally countable Borel combinatorics

…is the study of Borel combinatorial structures (e.g., trees, group actions) on standard Borel spaces (e.g., $\mathbb{R}, 2^{\mathbb{N}}$) which "live on" countable pieces.

Example A Borel action $\mathbb{F}_2 = \langle a, b \rangle \curvearrowright X$ induces a Schreier graph on X:

$$x \, G \, y \; :\Longleftrightarrow \; (ax = y) \vee (bx = y) \vee (ay = x) \vee (by = x)$$

If the action is free, then the graph is a forest (each component is a tree).

Many widely studied classes of locally countable Borel combinatorial structures:

- group actions
- graphs, trees, simplicial complexes, …
- graph colorings, perfect matchings, …

Locally countable Borel combinatorics

. . . is the study of Borel combinatorial structures (e.g., trees, group actions) on standard Borel spaces (e.g., $\mathbb{R}, 2^{\mathbb{N}}$) which "live on" countable pieces.

Example A Borel action $\mathbb{F}_2 = \langle a, b \rangle \curvearrowright X$ induces a Schreier graph on X:

$$x \; G \; y \;\; :\Longleftrightarrow \;\; (ax = y) \vee (bx = y) \vee (ay = x) \vee (by = x)$$

If the action is free, then the graph is a forest (each component is a tree).

Many widely studied classes of locally countable Borel combinatorial structures:

- group actions
- graphs, trees, simplicial complexes, . . .
- graph colorings, perfect matchings, . . .

This talk is about the *global* aspects of "all" locally ctbl Borel combinatorial structures.

Countable Borel equivalence relations (CBERs)

. . . $E \subseteq X^2$ are Borel equivalence relations with countable equivalence classes (the "countable pieces").

Instead of "Borel structures with countable pieces", we look at "Borel families of countable structures" on the classes of a CBER.

Countable Borel equivalence relations (CBERs)

… $E \subseteq X^2$ are Borel equivalence relations with countable equivalence classes (the "countable pieces").

Instead of "Borel structures with countable pieces", we look at "Borel families of countable structures" on the classes of a CBER.

Example If a CBER $E \subseteq X^2$ admits a Borel family of free *transitive* $\mathbb{F}_2$ actions on each class, then E also admits a Borel family of trees on each class.

Countable Borel equivalence relations (CBERs)

… $E \subseteq X^2$ are Borel equivalence relations with countable equivalence classes (the "countable pieces").

Instead of "Borel structures with countable pieces", we look at "Borel families of countable structures" on the classes of a CBER.

Example If a CBER $E \subseteq X^2$ admits a Borel family of free *transitive* $\mathbb{F}_2$ actions on each class, then E also admits a Borel family of trees on each class.

Example (Jackson–Kechris–Louveau) Turing equivalence $\equiv_T \subseteq (2^{\mathbb{N}})^2$ does *not* admit a Borel family of trees on each class.

Countable Borel equivalence relations (CBERs)

... $E \subseteq X^2$ are Borel equivalence relations with countable equivalence classes (the "countable pieces").

Instead of "Borel structures with countable pieces", we look at "Borel families of countable structures" on the classes of a CBER.

Example If a CBER $E \subseteq X^2$ admits a Borel family of free *transitive* $\mathbb{F}_2$ actions on each class, then E also admits a Borel family of trees on each class.

Example (Jackson–Kechris–Louveau) Turing equivalence $\equiv_T \subseteq (2^{\mathbb{N}})^2$ does *not* admit a Borel family of trees on each class.

Example (Feldman–Moore) *Every* CBER is induced by a Borel action of a countable group, i.e., admits a Borel family of transitive $\mathbb{F}_\omega = \langle g_0, g_1, \dots \rangle$ actions on each class.

Structurings

Let $\mathcal{L}$ be a countable first-order language.

Definition A **Borel $\mathcal{L}$-structuring** $\mathcal{M}$ of a CBER $E \subseteq X^2$ is a Borel family of countable $\mathcal{L}$-structures $(\mathcal{M}_C)_{C \in X/E}$ on each equivalcne class $C \in X/E$.

Structurings

Let $\mathcal{L}$ be a countable first-order language.

Definition A **Borel $\mathcal{L}$-structuring** $\mathcal{M}$ of a CBER $E \subseteq X^2$ is a Borel family of countable $\mathcal{L}$-structures $(\mathcal{M}_C)_{C \in X/E}$ on each equivalcne class $C \in X/E$.

Example A locally countable Borel graph $G \subseteq X^2$ is an $\mathcal{L}_{\mathsf{graph}}$-structuring for $\mathcal{L}_{\mathsf{graph}} = \{G\}$ (G a binary relation symbol) of any CBER $E \supseteq G$.

Structurings

Let $\mathcal{L}$ be a countable first-order language.

Definition A **Borel $\mathcal{L}$-structuring** $\mathcal{M}$ of a CBER $E \subseteq X^2$ is a Borel family of countable $\mathcal{L}$-structures $(\mathcal{M}_C)_{C \in X/E}$ on each equivalcne class $C \in X/E$.

Example A locally countable Borel graph $G \subseteq X^2$ is an $\mathcal{L}_{\mathsf{graph}}$-structuring for $\mathcal{L}_{\mathsf{graph}} = \{G\}$ (G a binary relation symbol) of any CBER $E \supseteq G$.

Example A Borel action of a countable group Γ generating E is an $\mathcal{L}_\Gamma$-structuring of E for $\mathcal{L}_\Gamma = \{a_\gamma\}_{\gamma \in \Gamma}$ (each a_γ a unary function symbol).

Structurings

Let $\mathcal{L}$ be a countable first-order language.

Definition A **Borel $\mathcal{L}$-structuring** $\mathcal{M}$ of a CBER $E \subseteq X^2$ is a Borel family of countable $\mathcal{L}$-structures $(\mathcal{M}_C)_{C \in X/E}$ on each equivalcne class $C \in X/E$.

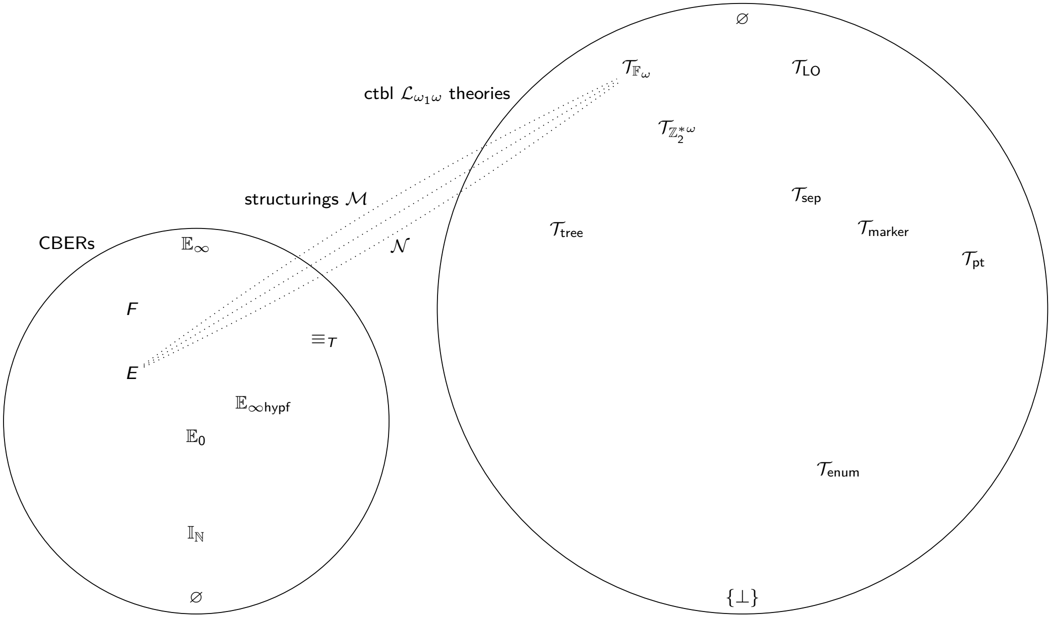
Example A locally countable Borel graph $G \subseteq X^2$ is an $\mathcal{L}_{\text{graph}}$-structuring for $\mathcal{L}_{\text{graph}} = \{G\}$ (G a binary relation symbol) of any CBER $E \supseteq G$.

Example A Borel action of a countable group Γ generating E is an $\mathcal{L}_\Gamma$-structuring of E for $\mathcal{L}_\Gamma = \{a_\gamma\}_{\gamma \in \Gamma}$ (each a_γ a unary function symbol).

Example A Borel Γ-action generating E is actually a structuring by models of

$$\mathcal{T}_\Gamma := \{\forall x\,(a_1(x) = x)\} \cup \{\forall x\,(a_\gamma(a_\delta(x)) = a_{\gamma\delta}(x)) \mid \gamma, \delta \in \Gamma\} \cup \Big\{\forall x, y \bigvee_{\gamma \in \Gamma} (a_\gamma(x) = y)\Big\}.$$

For a ctbl $\mathcal{L}_{\omega_1\omega}$ theory $\mathcal{T}$, a **$\mathcal{T}$-structuring** is an $\mathcal{L}$-structuring $\mathcal{M}$ s.t. each $\mathcal{M}_C \models \mathcal{T}$.



ctbl $\mathcal{L}_{\omega_1\omega}$ theories
$\varnothing$
$\mathcal{T}_{\mathbb{F}_\omega}$
$\mathcal{T}_{\mathsf{LO}}$
$\mathcal{T}_{\mathbb{Z}_2^{*\omega}}$
$\mathcal{T}_{\mathsf{sep}}$
$\mathcal{T}_{\mathsf{tree}}$
$\mathcal{T}_{\mathsf{marker}}$
$\mathcal{T}_{\mathsf{pt}}$
$\mathcal{T}_{\mathsf{enum}}$
$\{\bot\}$
structurings $\mathcal{M}$
$\mathcal{N}$
CBERs
$\mathbb{E}_\infty$
F
$\equiv_T$
E
$\mathbb{E}_{\infty\mathsf{hypf}}$
$\mathbb{E}_0$
$\mathbb{I}_{\mathbb{N}}$
$\varnothing$

Interpretations

Definition Let $(\mathcal{L}_1, \mathcal{T}_1)$, $(\mathcal{L}_2, \mathcal{T}_2)$ be ctbl $\mathcal{L}_{\omega_1\omega}$ theories (in relational languages).

An **interpretation** $\alpha : \mathcal{T}_1 \longrightarrow \mathcal{T}_2$

Interpretations

Definition Let $(\mathcal{L}_1, \mathcal{T}_1)$, $(\mathcal{L}_2, \mathcal{T}_2)$ be ctbl $\mathcal{L}_{\omega_1\omega}$ theories (in relational languages).

An **interpretation** $\alpha : \mathcal{T}_1 \longrightarrow \mathcal{T}_2$

is a mapping n-ary $\mathcal{L}_1$-relns $R \longmapsto \mathcal{L}_2$-formulas $\alpha(R)(x_0, \ldots, x_{n-1})$

Interpretations

Definition Let $(\mathcal{L}_1, \mathcal{T}_1)$, $(\mathcal{L}_2, \mathcal{T}_2)$ be ctbl $\mathcal{L}_{\omega_1\omega}$ theories (in relational languages).

An **interpretation** $\alpha : \mathcal{T}_1 \longrightarrow \mathcal{T}_2$

is a mapping n-ary $\mathcal{L}_1$-relns $R \longmapsto \mathcal{L}_2$-formulas $\alpha(R)(x_0, \ldots, x_{n-1})$

such that these formulas define a model of $\mathcal{T}_1$ in every model of $\mathcal{T}_2$:

$$\begin{aligned} \mathrm{Mod}(\mathcal{T}_1) &\longleftarrow \mathrm{Mod}(\mathcal{T}_2) \\ (\alpha(R)^{\mathcal{M}})_{R \in \mathcal{L}_1} &\longleftarrow\!\!\shortmid \mathcal{M}. \end{aligned}$$

Interpretations

Definition Let $(\mathcal{L}_1, \mathcal{T}_1)$, $(\mathcal{L}_2, \mathcal{T}_2)$ be ctbl $\mathcal{L}_{\omega_1\omega}$ theories (in relational languages).

An **interpretation** $\alpha : \mathcal{T}_1 \longrightarrow \mathcal{T}_2$

is a mapping n-ary $\mathcal{L}_1$-relns $R \longmapsto \mathcal{L}_2$-formulas $\alpha(R)(x_0, \ldots, x_{n-1})$

such that these formulas define a model of $\mathcal{T}_1$ in every model of $\mathcal{T}_2$:

$$\mathrm{Mod}(\mathcal{T}_1) \longleftarrow \mathrm{Mod}(\mathcal{T}_2)$$

$$(\alpha(R)^{\mathcal{M}})_{R \in \mathcal{L}_1} \longleftarrow\!\!\dashv \mathcal{M}.$$

Example Given an $\mathbb{F}_\omega$-action $\mathcal{M} = (C, a_\gamma)_{\gamma \in \mathbb{F}_\omega = \langle g_0, g_1, \ldots \rangle}$, we have the Schreier graph

$$x \; G \; y \;\; :\Longleftrightarrow \;\; \bigvee_n ((a_{g_n}(x) = y) \vee (x = a_{g_n}(y))) \, .$$

Interpretations

Definition Let $(\mathcal{L}_1, \mathcal{T}_1)$, $(\mathcal{L}_2, \mathcal{T}_2)$ be ctbl $\mathcal{L}_{\omega_1\omega}$ theories (in relational languages).

An **interpretation** $\alpha : \mathcal{T}_1 \longrightarrow \mathcal{T}_2$

is a mapping $n\text{-ary } \mathcal{L}_1\text{-relns } R \longmapsto \mathcal{L}_2\text{-formulas } \alpha(R)(x_0, \ldots, x_{n-1})$

such that these formulas define a model of $\mathcal{T}_1$ in every model of $\mathcal{T}_2$:

$$\begin{aligned} \mathrm{Mod}(\mathcal{T}_1) &\longleftarrow \mathrm{Mod}(\mathcal{T}_2) \\ (\alpha(R)^{\mathcal{M}})_{R \in \mathcal{L}_1} &\longleftarrow\!\shortmid \mathcal{M}. \end{aligned}$$

Example Given an $\mathbb{F}_\omega$-action $\mathcal{M} = (C, a_\gamma)_{\gamma \in \mathbb{F}_\omega = \langle g_0, g_1, \ldots \rangle}$, we have the Schreier graph

$$x \; G \; y \;\; :\Longleftrightarrow \;\; \underbrace{\bigvee_n ((a_{g_n}(x) = y) \vee (x = a_{g_n}(y)))}_{\alpha(G)(x,y)}.$$

This is defined by an interpretation

α : theory of graphs (in $\mathcal{L}_{\text{graph}} = \{G\}$) $\longrightarrow$ theory of $\mathbb{F}_\omega$-actions (in $\mathcal{L}_{\mathbb{F}_\omega} = \{a_\gamma\}_{\gamma \in \mathbb{F}_\omega}$).

Interpretations

Definition Let $(\mathcal{L}_1, \mathcal{T}_1)$, $(\mathcal{L}_2, \mathcal{T}_2)$ be ctbl $\mathcal{L}_{\omega_1\omega}$ theories (in relational languages).

An **interpretation** $\alpha : \mathcal{T}_1 \longrightarrow \mathcal{T}_2$

is a mapping $n\text{-ary } \mathcal{L}_1\text{-relns } R \longmapsto \mathcal{L}_2\text{-formulas } \alpha(R)(x_0, \dots, x_{n-1})$

such that these formulas define a model of $\mathcal{T}_1$ in every model of $\mathcal{T}_2$:

$$\begin{aligned} \mathrm{Mod}(\mathcal{T}_1) &\longleftarrow \mathrm{Mod}(\mathcal{T}_2) \\ (\alpha(R)^{\mathcal{M}})_{R \in \mathcal{L}_1} &\longleftarrow\!\!\shortmid \mathcal{M}. \end{aligned}$$

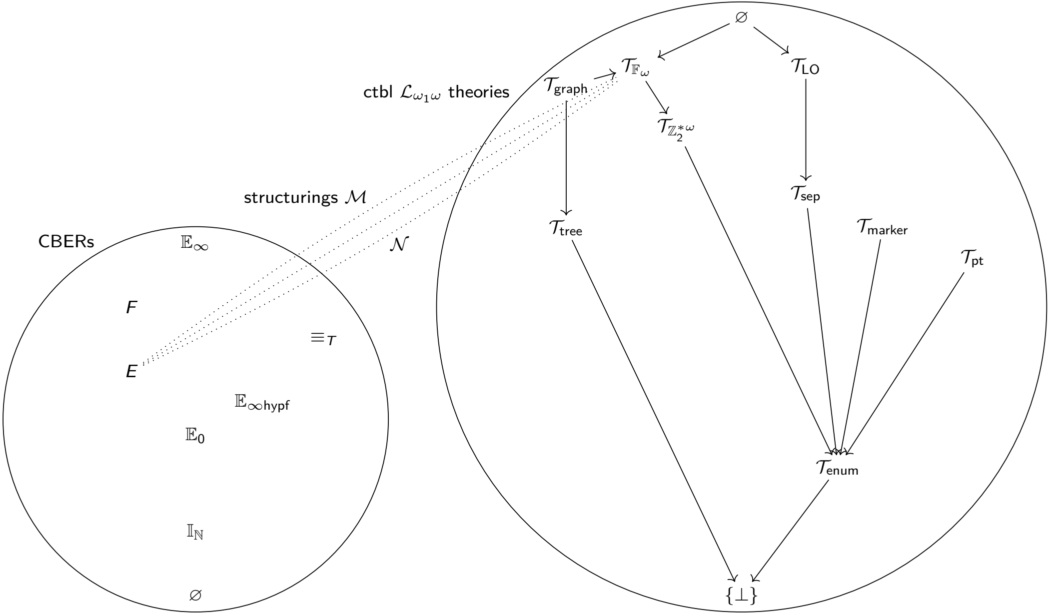
Example Given an $\mathbb{F}_\omega$-action $\mathcal{M} = (C, a_\gamma)_{\gamma \in \mathbb{F}_\omega = \langle g_0, g_1, \dots \rangle}$, we have the Schreier graph

$$x \; G \; y \quad :\Longleftrightarrow \quad \underbrace{\bigvee_n ((a_{g_n}(x) = y) \vee (x = a_{g_n}(y)))}_{\alpha(G)(x,y)}.$$

This is defined by an interpretation

α : theory of graphs (in $\mathcal{L}_{\text{graph}} = \{G\}$) $\longrightarrow$ theory of $\mathbb{F}_\omega$-actions (in $\mathcal{L}_{\mathbb{F}_\omega} = \{a_\gamma\}_{\gamma \in \mathbb{F}_\omega}$).

Note There is a more general model-theoretic notion of "imaginary interpretation" that we're not using.


CBERs
$\mathbb{E}_\infty$
F
$\equiv_T$
E
$\mathbb{E}_{\infty\mathsf{hypf}}$
$\mathbb{E}_0$
$\mathbb{I}_\mathbb{N}$
$\varnothing$
structurings $\mathcal{M}$
$\mathcal{N}$
ctbl $\mathcal{L}_{\omega_1\omega}$ theories
$\varnothing$
$\mathcal{T}_{\mathbb{F}_\omega}$
$\mathcal{T}_{\mathsf{LO}}$
$\mathcal{T}_{\mathsf{graph}}$
$\mathcal{T}_{\mathbb{Z}_2^{*\omega}}$
$\mathcal{T}_{\mathsf{sep}}$
$\mathcal{T}_{\mathsf{tree}}$
$\mathcal{T}_{\mathsf{marker}}$
$\mathcal{T}_{\mathsf{pt}}$
$\mathcal{T}_{\mathsf{enum}}$
$\{\bot\}$

Interpretations and structurings

An **interpretation** $\alpha : \mathcal{T}_1 \longrightarrow \mathcal{T}_2$

yields a mapping $\mathrm{Mod}(\mathcal{T}_1) \longleftarrow \mathrm{Mod}(\mathcal{T}_2)$

hence also given a CBER $E \subseteq X^2$,

$$\{\mathcal{T}_1\text{-structurings of } E\} \longleftarrow \{\mathcal{T}_2\text{-structurings of } E\}.$$

Interpretations and structurings

An **interpretation** $\alpha : \mathcal{T}_1 \longrightarrow \mathcal{T}_2$

yields a mapping $\mathrm{Mod}(\mathcal{T}_1) \longleftarrow \mathrm{Mod}(\mathcal{T}_2)$

hence also given a CBER $E \subseteq X^2$,

$$\{\mathcal{T}_1\text{-structurings of } E\} \longleftarrow \{\mathcal{T}_2\text{-structurings of } E\}.$$

Example There is an interpretation $\alpha : \mathcal{T}_{\text{4-reg tree}} \to \mathcal{T}_{\text{free } \mathbb{F}_2\text{-action}}$.
Thus, every CBER E induced by a free $\mathbb{F}_2$-action admits a structuring by 4-reg trees.

Interpretations and structurings

An **interpretation** $\alpha : \mathcal{T}_1 \longrightarrow \mathcal{T}_2$

yields a mapping $\mathrm{Mod}(\mathcal{T}_1) \longleftarrow \mathrm{Mod}(\mathcal{T}_2)$

hence also given a CBER $E \subseteq X^2$,

$$\{\mathcal{T}_1\text{-structurings of } E\} \longleftarrow \{\mathcal{T}_2\text{-structurings of } E\}.$$

Example There is an interpretation $\alpha : \mathcal{T}_{\text{4-reg tree}} \to \mathcal{T}_{\text{free } \mathbb{F}_2\text{-action}}$.
Thus, every CBER E induced by a free $\mathbb{F}_2$-action admits a structuring by 4-reg trees.

The converse is also true.
However, there is obviously no interpretation $\mathcal{T}_{\text{free } \mathbb{F}_2\text{-action}} \to \mathcal{T}_{\text{4-reg tree}}$!

Interpretations and structurings

An **interpretation** $\alpha : \mathcal{T}_1 \longrightarrow \mathcal{T}_2$

yields a mapping $\mathrm{Mod}(\mathcal{T}_1) \longleftarrow \mathrm{Mod}(\mathcal{T}_2)$

hence also given a CBER $E \subseteq X^2$,

$$\{\mathcal{T}_1\text{-structurings of } E\} \longleftarrow \{\mathcal{T}_2\text{-structurings of } E\}.$$

Example There is an interpretation $\alpha : \mathcal{T}_{\text{4-reg tree}} \to \mathcal{T}_{\text{free } \mathbb{F}_2\text{-action}}$.
Thus, every CBER E induced by a free $\mathbb{F}_2$-action admits a structuring by 4-reg trees.

The converse is also true.
However, there is obviously no interpretation $\mathcal{T}_{\text{free } \mathbb{F}_2\text{-action}} \to \mathcal{T}_{\text{4-reg tree}}$!

Example (Feldman–Moore) *Every* CBER (structured by $\varnothing$) is structurable by $\mathcal{T}_{\mathbb{F}_\omega}$.
There is obviously no interpretation $\mathcal{T}_{\mathbb{F}_\omega} \to \varnothing$!

Freely available structures on CBERs

The most important theorem in locally countable Borel combinatorics:

Theorem (Lusin–Novikov)

Every CB(E)R $E \subseteq X^2$ *can be written as* $E = \bigcup_n f_n$ *for Borel* $f_n : X \to X$.

Freely available structures on CBERs

The most important theorem in locally countable Borel combinatorics:

Theorem (Lusin–Novikov)

Every CB(E)R $E \subseteq X^2$ can be written as $E = \bigcup_n f_n$ for Borel $f_n : X \to X$.

In other words, every CBER admits a structuring by

$$\mathcal{T}_{\mathrm{LN}} := \{\forall x, y \bigvee_n (f_n(x) = y)\}, \text{ in the language } \mathcal{L}_{\mathrm{LN}} := \{f_n\}_{n \in \mathbb{N}}.$$

Freely available structures on CBERs

The most important theorem in locally countable Borel combinatorics:

Theorem (Lusin–Novikov)

Every CB(E)R $E \subseteq X^2$ can be written as $E = \bigcup_n f_n$ for Borel $f_n : X \to X$.

In other words, every CBER admits a structuring by

$$\mathcal{T}_{\mathrm{LN}} := \{\forall x, y \bigvee_n (f_n(x) = y)\}, \text{ in the language } \mathcal{L}_{\mathrm{LN}} := \{f_n\}_{n \in \mathbb{N}}.$$

By second-countability of X, every CBER is also structurable by

$$\mathcal{T}_{\mathrm{sep}} := \{\forall x \neq y \bigvee_k (U_k(x) \leftrightarrow \neg U_k(y))\}, \text{ in language } \mathcal{L}_{\mathrm{sep}} := \{U_k\}_{k \in \mathbb{N}}.$$

Freely available structures on CBERs

The most important theorem in locally countable Borel combinatorics:

Theorem (Lusin–Novikov)

Every CB(E)R $E \subseteq X^2$ can be written as $E = \bigcup_n f_n$ for Borel $f_n : X \to X$.

In other words, every CBER admits a structuring by

$$\mathcal{T}_{\mathsf{LN}} := \{\forall x, y \bigvee_n (f_n(x) = y)\}, \text{ in the language } \mathcal{L}_{\mathsf{LN}} := \{f_n\}_{n \in \mathbb{N}}.$$

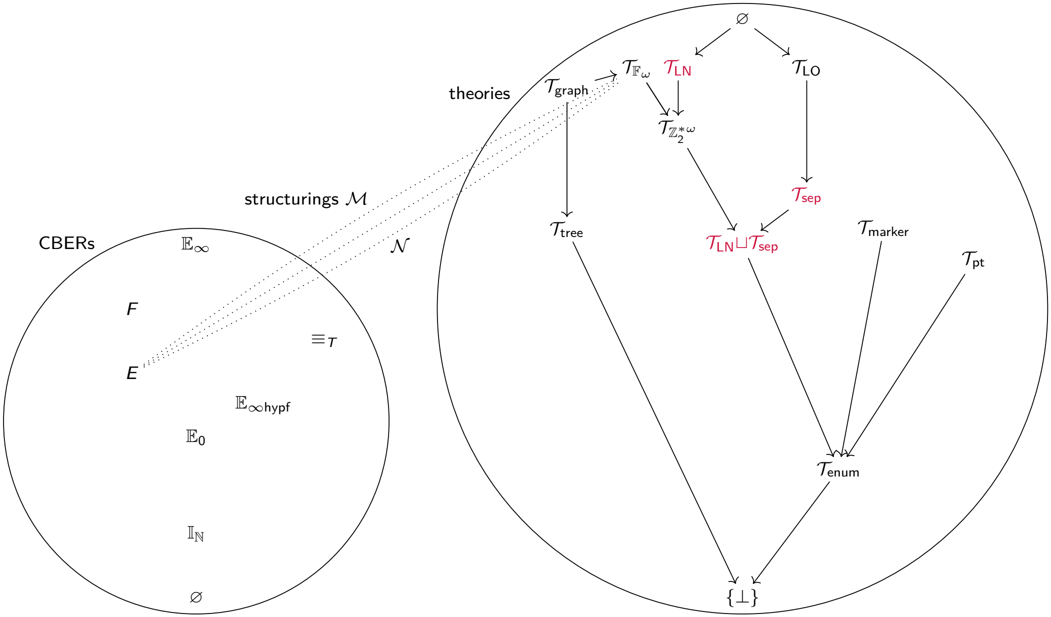
By second-countability of X, every CBER is also structurable by

$$\mathcal{T}_{\mathsf{sep}} := \{\forall x \neq y \bigvee_k (U_k(x) \leftrightarrow \neg U_k(y))\}, \text{ in language } \mathcal{L}_{\mathsf{sep}} := \{U_k\}_{k \in \mathbb{N}}.$$

Example We have an interpretation $\mathcal{T}_{\mathsf{LO}} \to \mathcal{T}_{\mathsf{sep}}$ given by

$$x < y \;:\Longleftrightarrow\; \bigvee_n \Big(\bigwedge_{k<n} (U_k(x) \leftrightarrow U_k(y)) \wedge \neg U_k(x) \wedge U_k(y) \Big),$$

hence every CBER admits a linear order on each class.



CBERs
$\mathbb{E}_\infty$
F
$\equiv_T$
E
$\mathbb{E}_{\infty\mathrm{hypf}}$
$\mathbb{E}_0$
$\mathbb{I}_\mathbb{N}$
$\varnothing$
structurings $\mathcal{M}$
$\mathcal{N}$
theories
$\varnothing$
$\mathcal{T}_{\mathrm{graph}}$
$\mathcal{T}_{\mathbb{F}_\omega}$
$\mathcal{T}_{\mathrm{LN}}$
$\mathcal{T}_{\mathrm{LO}}$
$\mathcal{T}_{\mathbb{Z}_2^{*\omega}}$
$\mathcal{T}_{\mathrm{sep}}$
$\mathcal{T}_{\mathrm{tree}}$
$\mathcal{T}_{\mathrm{LN}} \sqcup \mathcal{T}_{\mathrm{sep}}$
$\mathcal{T}_{\mathrm{marker}}$
$\mathcal{T}_{\mathrm{pt}}$
$\mathcal{T}_{\mathrm{enum}}$
$\{\bot\}$

CBERs $\hookrightarrow$ theories

Theorem (C.–Kechris 2018, Banerjee–C. 2024)

We have a canonical assignment

$$\{CBERs\} \hookrightarrow \{ctbl\ \mathcal{L}_{\omega_1\omega}\ theories\}$$
$$E \longmapsto \mathcal{T}_E$$

of a theory $\mathcal{T}_E$ to each CBER E, called its ***Scott theory****, such that*

CBERs $\hookrightarrow$ theories

Theorem (C.–Kechris 2018, Banerjee–C. 2024)

We have a canonical assignment

$$\{CBERs\} \longhookrightarrow \{ctbl\ \mathcal{L}_{\omega_1\omega}\ theories\}$$
$$E \longmapsto \mathcal{T}_E$$

of a theory $\mathcal{T}_E$ to each CBER E, called its ***Scott theory****, such that*

(a) *For any other theory $\mathcal{T}$, $\{\mathcal{T}$-structurings of $E\} \cong \{$interpretations $\mathcal{T} \to \mathcal{T}_E\}$.*

CBERs $\hookrightarrow$ theories

Theorem (C.–Kechris 2018, Banerjee–C. 2024)

We have a canonical assignment

$$\{\text{CBERs}\} \hookrightarrow \{\text{ctbl } \mathcal{L}_{\omega_1\omega} \text{ theories}\}$$
$$E \longmapsto \mathcal{T}_E$$

of a theory $\mathcal{T}_E$ *to each CBER* E*, called its* ***Scott theory****, such that*

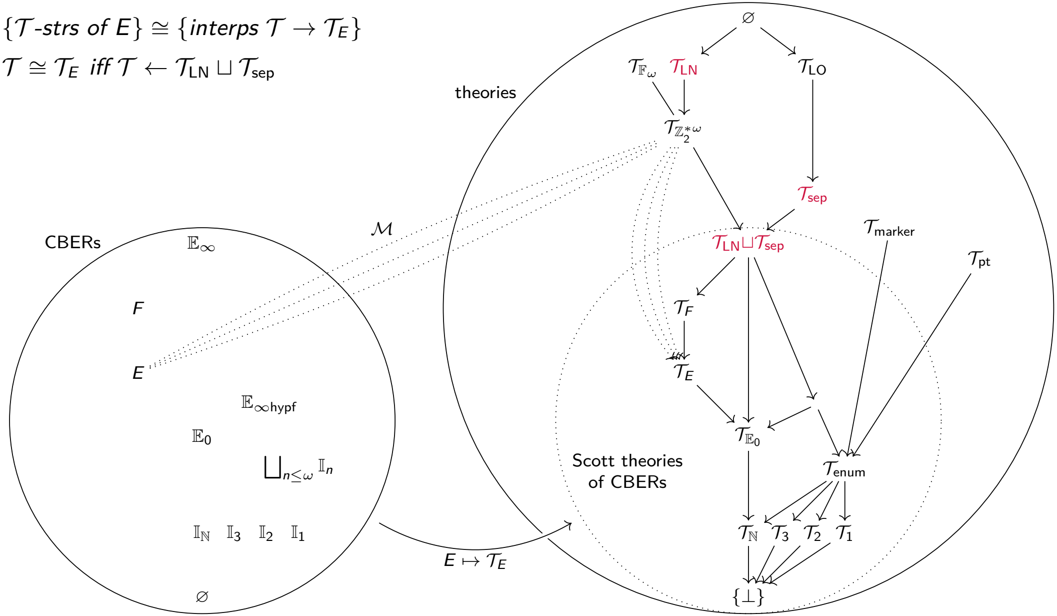
(a) *For any other theory* $\mathcal{T}$, $\{\mathcal{T}\text{-structurings of } E\} \cong \{\text{interpretations } \mathcal{T} \to \mathcal{T}_E\}$.

(b) *Up to bi-interpretations, the theories* $\mathcal{T}_E$ *are precisely those s.t.* $\mathcal{T}_E \leftarrow \mathcal{T}_{\mathsf{LN}} \sqcup \mathcal{T}_{\mathsf{sep}}$.

Main Theorem

(a) $\{\mathcal{T}\text{-strs of } E\} \cong \{\text{interps } \mathcal{T} \to \mathcal{T}_E\}$

(b) $\mathcal{T} \cong \mathcal{T}_E$ iff $\mathcal{T} \leftarrow \mathcal{T}_{\mathrm{LN}} \sqcup \mathcal{T}_{\mathrm{sep}}$



CBERs $\hookrightarrow$ theories

Theorem (C.–Kechris 2018, Banerjee–C. 2024)

We have a canonical assignment

$$\begin{aligned} \{\text{CBERs}\} &\hookrightarrow \{\text{ctbl } \mathcal{L}_{\omega_1\omega} \text{ theories}\} \\ E &\longmapsto \mathcal{T}_E \end{aligned}$$

of a theory $\mathcal{T}_E$ to each CBER E, called its ***Scott theory****, such that*

(a) *For any other theory $\mathcal{T}$, $\{\mathcal{T}\text{-structurings of } E\} \cong \{\text{interpretations } \mathcal{T} \to \mathcal{T}_E\}$.*

(b) *Up to bi-interpretations, the theories $\mathcal{T}_E$ are precisely those s.t. $\mathcal{T}_E \leftarrow \mathcal{T}_{\mathsf{LN}} \sqcup \mathcal{T}_{\mathsf{sep}}$.*

Corollary (folklore, Banerjee–C.)

For any theories $\mathcal{T}_1, \mathcal{T}_2$, the following are equivalent:

(a) *Every CBER E admitting a $\mathcal{T}_2$-structuring also admits a $\mathcal{T}_1$-structuring.*

(b) *There exists an interpretation $\mathcal{T}_1 \to \mathcal{T}_2 \sqcup \mathcal{T}_{\mathsf{LN}} \sqcup \mathcal{T}_{\mathsf{sep}}$.*

CBERs $\hookrightarrow$ theories

Theorem (C.–Kechris 2018, Banerjee–C. 2024)

We have a canonical assignment

$$\begin{aligned}\{\mathit{CBERs}\} &\hookrightarrow \{\mathit{ctbl}\ \mathcal{L}_{\omega_1\omega}\ \mathit{theories}\} \\ E &\longmapsto \mathcal{T}_E\end{aligned}$$

of a theory $\mathcal{T}_E$ to each CBER E, called its ***Scott theory****, such that*

(a) *For any other theory $\mathcal{T}$, $\{\mathcal{T}$-structurings of $E\} \cong \{$interpretations $\mathcal{T} \to \mathcal{T}_E\}$.*

(b) *Up to bi-interpretations, the theories $\mathcal{T}_E$ are precisely those s.t. $\mathcal{T}_E \leftarrow \mathcal{T}_{\mathrm{LN}} \sqcup \mathcal{T}_{\mathrm{sep}}$.*

Corollary (folklore, Banerjee–C.)

For any theories $\mathcal{T}_1, \mathcal{T}_2$, the following are equivalent:

(a) *Every theory $\mathcal{T}_E \leftarrow \mathcal{T}_2 \sqcup \mathcal{T}_{\mathrm{LN}} \sqcup \mathcal{T}_{\mathrm{sep}}$ also admits $\mathcal{T}_E \leftarrow \mathcal{T}_1$.*

(b) *There exists an interpretation $\mathcal{T}_1 \to \mathcal{T}_2 \sqcup \mathcal{T}_{\mathrm{LN}} \sqcup \mathcal{T}_{\mathrm{sep}}$.*

CBERs $\hookrightarrow$ theories

Theorem (C.–Kechris 2018, Banerjee–C. 2024)

We have a canonical assignment

$$\begin{aligned} \{\mathit{CBERs}\} &\longhookrightarrow \{\mathit{ctbl}\ \mathcal{L}_{\omega_1\omega}\ \mathit{theories}\} \\ E &\longmapsto \mathcal{T}_E \end{aligned}$$

of a theory $\mathcal{T}_E$ to each CBER E, called its ***Scott theory****, such that*

(a) *For any other theory $\mathcal{T}$, $\{\mathcal{T}$-structurings of $E\} \cong \{$interpretations $\mathcal{T} \to \mathcal{T}_E\}$.*

(b) *Up to bi-interpretations, the theories $\mathcal{T}_E$ are precisely those s.t. $\mathcal{T}_E \leftarrow \mathcal{T}_{\mathsf{LN}} \sqcup \mathcal{T}_{\mathsf{sep}}$.*

For a CBER $E \subseteq X^2$, we define $\mathcal{T}_E$ by declaring that models $\mathcal{M} = (Y, \dots)$ of $\mathcal{T}_E$ on a countable set Y to be bijections $Y \to X$ onto an E-class.

CBERs $\hookrightarrow$ theories

Theorem (C.–Kechris 2018, Banerjee–C. 2024)

We have a canonical assignment

$$\{CBERs\} \hookrightarrow \{ctbl\ \mathcal{L}_{\omega_1\omega}\ theories\}$$
$$E \longmapsto \mathcal{T}_E$$

*of a theory $\mathcal{T}_E$ to each CBER E, called its **Scott theory**, such that*

(a) *For any other theory $\mathcal{T}$, $\{\mathcal{T}$-structurings of $E\} \cong \{$interpretations $\mathcal{T} \to \mathcal{T}_E\}$.*

(b) *Up to bi-interpretations, the theories $\mathcal{T}_E$ are precisely those s.t. $\mathcal{T}_E \leftarrow \mathcal{T}_{\mathrm{LN}} \sqcup \mathcal{T}_{\mathrm{sep}}$.*

For a CBER $E \subseteq X^2$, we define $\mathcal{T}_E$ by declaring that models $\mathcal{M} = (Y, \dots)$ of $\mathcal{T}_E$ on a countable set Y to be bijections $Y \to X$ onto an E-class.

Given $\mathcal{T} \leftarrow \mathcal{T}_{\mathrm{LN}} \sqcup \mathcal{T}_{\mathrm{sep}}$, we have $\mathcal{T} \cong \mathcal{T}_E$ for E on $X = \mathcal{S}_1(\mathcal{T})$, where two 1-types are E-related iff they are realized in the same model.

CBERs $\hookrightarrow$ theories

Theorem (C.–Kechris 2018, Banerjee–C. 2024)

We have a canonical assignment

$$\begin{aligned} \{\textit{CBERs}\} &\longhookrightarrow \{\textit{ctbl } \mathcal{L}_{\omega_1\omega} \textit{ theories}\} \\ E &\longmapsto \mathcal{T}_E \end{aligned}$$

of a theory $\mathcal{T}_E$ to each CBER E, called its ***Scott theory****, such that*

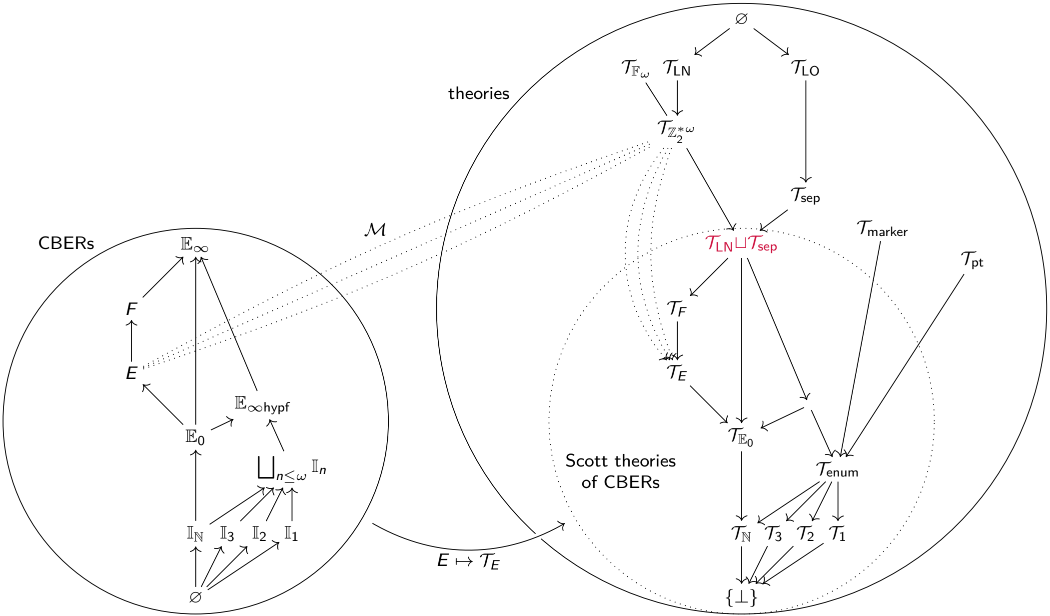
(a) *For any other theory $\mathcal{T}$, $\{\mathcal{T}$-structurings of $E\} \cong \{$interpretations $\mathcal{T} \to \mathcal{T}_E\}$.*

(b) *Up to bi-interpretations, the theories $\mathcal{T}_E$ are precisely those s.t. $\mathcal{T}_E \leftarrow \mathcal{T}_{\mathsf{LN}} \sqcup \mathcal{T}_{\mathsf{sep}}$.*

For a CBER $E \subseteq X^2$, we define $\mathcal{T}_E$ by declaring that models $\mathcal{M} = (Y, \dots)$ of $\mathcal{T}_E$ on a countable set Y to be bijections $Y \to X$ onto an E-class.

Note that for two CBERs $E \subseteq X$ and $F \subseteq Y$,

$$\begin{aligned} \{\text{interps } \mathcal{T}_E \to \mathcal{T}_F\} &\cong \{\mathcal{T}_E\text{-structurings of } F\} \\ &\cong \{\text{Borel } \textbf{class-bijective homomorphisms } (Y, F) \to (X, E)\}. \end{aligned}$$


CBERs
$\mathbb{E}_\infty$
F
E
$\mathbb{E}_{\infty\mathsf{hypf}}$
$\mathbb{E}_0$
$\bigsqcup_{n\leq\omega} \mathbb{I}_n$
$\mathbb{I}_\mathbb{N}$ $\mathbb{I}_3$ $\mathbb{I}_2$ $\mathbb{I}_1$
$\varnothing$
$\mathcal{M}$
$E \mapsto \mathcal{T}_E$
theories
$\varnothing$
$\mathcal{T}_{\mathbb{F}_\omega}$ $\mathcal{T}_{\mathsf{LN}}$ $\mathcal{T}_{\mathsf{LO}}$
$\mathcal{T}_{\mathbb{Z}_2^{*\omega}}$
$\mathcal{T}_{\mathsf{sep}}$
$\mathcal{T}_{\mathsf{LN}} \sqcup \mathcal{T}_{\mathsf{sep}}$
$\mathcal{T}_{\mathsf{marker}}$
$\mathcal{T}_{\mathsf{pt}}$
$\mathcal{T}_F$
$\mathcal{T}_E$
$\mathcal{T}_{\mathbb{E}_0}$
$\mathcal{T}_{\mathsf{enum}}$
Scott theories
of CBERs
$\mathcal{T}_\mathbb{N}$ $\mathcal{T}_3$ $\mathcal{T}_2$ $\mathcal{T}_1$
$\{\bot\}$

Interpretations in the wild

Corollary (folklore, Banerjee–C.)

For any theories $\mathcal{T}_1, \mathcal{T}_2$, the following are equivalent:

(a) *Every CBER E admitting a $\mathcal{T}_2$-structuring also admits a $\mathcal{T}_1$-structuring.*

(b) *There exists an interpretation $\mathcal{T}_1 \to \mathcal{T}_2 \sqcup \mathcal{T}_{\mathsf{LN}} \sqcup \mathcal{T}_{\mathsf{sep}}$.*

Interpretations in the wild

Corollary (folklore, Banerjee–C.)

For any theories $\mathcal{T}_1, \mathcal{T}_2$, the following are equivalent:

(a) *Every CBER E admitting a $\mathcal{T}_2$-structuring also admits a $\mathcal{T}_1$-structuring.*

(b) *There exists an interpretation $\mathcal{T}_1 \to \mathcal{T}_2 \sqcup \mathcal{T}_{\mathsf{LN}} \sqcup \mathcal{T}_{\mathsf{sep}}$.*

In practice, many theorems of the form (b) are essentially proved via (a).

Example (Feldman–Moore) $\mathcal{T}_{\mathbb{F}_\omega} =$ "transitive $\mathbb{F}_\omega$-actions" $\to \mathcal{T}_{\mathsf{LN}} \sqcup \mathcal{T}_{\mathsf{sep}}$.
In fact, $\mathcal{T}_{\mathbb{Z}_2^{*\omega}} \to \mathcal{T}_{\mathsf{LN}} \sqcup \mathcal{T}_{\mathsf{sep}}$.
In fact, $\mathcal{T}_{\mathsf{color}_2} =$ "ω-colorings of complete graph" $\to \mathcal{T}_{\mathsf{LN}} \sqcup \mathcal{T}_{\mathsf{sep}}$.

Interpretations in the wild

Corollary (folklore, Banerjee–C.)

For any theories $\mathcal{T}_1, \mathcal{T}_2$, the following are equivalent:

(a) *Every CBER E admitting a $\mathcal{T}_2$-structuring also admits a $\mathcal{T}_1$-structuring.*

(b) *There exists an interpretation $\mathcal{T}_1 \to \mathcal{T}_2 \sqcup \mathcal{T}_{\mathsf{LN}} \sqcup \mathcal{T}_{\mathsf{sep}}$.*

In practice, many theorems of the form (b) are essentially proved via (a).

Example (Feldman–Moore) $\mathcal{T}_{\mathbb{F}_\omega}$ = "transitive $\mathbb{F}_\omega$-actions" $\to \mathcal{T}_{\mathsf{LN}} \sqcup \mathcal{T}_{\mathsf{sep}}$.
In fact, $\mathcal{T}_{\mathbb{Z}_2^{*\omega}} \to \mathcal{T}_{\mathsf{LN}} \sqcup \mathcal{T}_{\mathsf{sep}}$.
In fact, $\mathcal{T}_{\mathsf{color}_2}$ = "ω-colorings of complete graph" $\to \mathcal{T}_{\mathsf{LN}} \sqcup \mathcal{T}_{\mathsf{sep}}$.

Example (Kechris–Miller)
$\mathcal{T}_{\mathsf{color}_{<\omega}}$ = "ω-colorings of complete $< \omega$-hypergraph" $\to \mathcal{T}_{\mathsf{LN}} \sqcup \mathcal{T}_{\mathsf{sep}}$.

Interpretations in the wild

Corollary (folklore, Banerjee–C.)

For any theories $\mathcal{T}_1, \mathcal{T}_2$, the following are equivalent:

(a) *Every CBER E admitting a $\mathcal{T}_2$-structuring also admits a $\mathcal{T}_1$-structuring.*

(b) *There exists an interpretation $\mathcal{T}_1 \to \mathcal{T}_2 \sqcup \mathcal{T}_{\mathsf{LN}} \sqcup \mathcal{T}_{\mathsf{sep}}$.*

In practice, many theorems of the form (b) are essentially proved via (a).

Example (Feldman–Moore) $\mathcal{T}_{\mathbb{F}_\omega}$ = "transitive $\mathbb{F}_\omega$-actions" $\to \mathcal{T}_{\mathsf{LN}} \sqcup \mathcal{T}_{\mathsf{sep}}$.
In fact, $\mathcal{T}_{\mathbb{Z}_2^{*\omega}} \to \mathcal{T}_{\mathsf{LN}} \sqcup \mathcal{T}_{\mathsf{sep}}$.
In fact, $\mathcal{T}_{\mathsf{color}_2}$ = "ω-colorings of complete graph" $\to \mathcal{T}_{\mathsf{LN}} \sqcup \mathcal{T}_{\mathsf{sep}}$.

Example (Kechris–Miller)
$\mathcal{T}_{\mathsf{color}_{<\omega}}$ = "ω-colorings of complete $<\omega$-hypergraph" $\to \mathcal{T}_{\mathsf{LN}} \sqcup \mathcal{T}_{\mathsf{sep}}$.

Example (Slaman–Steel) $\mathcal{T}_{\mathsf{marker}}$ = "$\Downarrow\!\bigcap_n A_n = \varnothing$, $A_n \neq \varnothing$" $\to \mathcal{T}_{\mathsf{sep}} \sqcup \mathcal{T}_{\mathsf{inf}}$.

Interpretations in the wild

Example (Feldman–Moore) $\mathcal{T}_{\mathbb{F}_\omega} =$ "transitive $\mathbb{F}_\omega$-actions" $\to \mathcal{T}_{\mathsf{LN}} \sqcup \mathcal{T}_{\mathsf{sep}}$.
In fact, $\mathcal{T}_{\mathbb{Z}_2^{*\omega}} \to \mathcal{T}_{\mathsf{LN}} \sqcup \mathcal{T}_{\mathsf{sep}}$.
In fact, $\mathcal{T}_{\mathsf{color}_2} =$ "ω-colorings of complete graph" $\to \mathcal{T}_{\mathsf{LN}} \sqcup \mathcal{T}_{\mathsf{sep}}$.

Example (Kechris–Miller)
$\mathcal{T}_{\mathsf{color}_{<\omega}} =$ "ω-colorings of complete $< \omega$-hypergraph" $\to \mathcal{T}_{\mathsf{LN}} \sqcup \mathcal{T}_{\mathsf{sep}}$.

Example (Slaman–Steel) $\mathcal{T}_{\mathsf{marker}} =$ "$\bigcap\!\!\!\downarrow_n A_n = \varnothing,\ A_n \neq \varnothing$" $\to \mathcal{T}_{\mathsf{sep}} \sqcup \mathcal{T}_{\mathsf{inf}}$.

Theorem (Banerjee-C.) *None of the interpretabilities*

$$\mathcal{T}_{\mathbb{F}_\omega} \longrightarrow \mathcal{T}_{\mathbb{Z}_2^{*\omega}} \longrightarrow \mathcal{T}_{\mathsf{color}_2} \longrightarrow \mathcal{T}_{\mathsf{color}_{<\omega}} \longrightarrow \mathcal{T}_{\mathsf{LN}} \sqcup \mathcal{T}_{\mathsf{sep}}$$

can be reversed.

Interpretations in the wild

Example (Feldman–Moore) $\mathcal{T}_{\mathbb{F}_\omega}$ = "transitive $\mathbb{F}_\omega$-actions" $\to \mathcal{T}_{\mathsf{LN}} \sqcup \mathcal{T}_{\mathsf{sep}}$.
In fact, $\mathcal{T}_{\mathbb{Z}_2^{*\omega}} \to \mathcal{T}_{\mathsf{LN}} \sqcup \mathcal{T}_{\mathsf{sep}}$.
In fact, $\mathcal{T}_{\mathsf{color}_2}$ = "ω-colorings of complete graph" $\to \mathcal{T}_{\mathsf{LN}} \sqcup \mathcal{T}_{\mathsf{sep}}$.

Example (Kechris–Miller)
$\mathcal{T}_{\mathsf{color}_{<\omega}}$ = "ω-colorings of complete $<\omega$-hypergraph" $\to \mathcal{T}_{\mathsf{LN}} \sqcup \mathcal{T}_{\mathsf{sep}}$.

Example (Slaman–Steel) $\mathcal{T}_{\mathsf{marker}}$ = "$\bigcap\!\!\!\downarrow_n A_n = \varnothing$, $A_n \neq \varnothing$" $\to \mathcal{T}_{\mathsf{sep}} \sqcup \mathcal{T}_{\mathsf{inf}}$.

Theorem (Banerjee-C.) *None of the interpretabilities*

$$\underbrace{\mathcal{T}_{\mathsf{LN}} \rightleftarrows \mathcal{T}_{\mathbb{F}_\omega} \longrightarrow \mathcal{T}_{\mathbb{Z}_2^{*\omega}} \longrightarrow \mathcal{T}_{\mathsf{color}_2}}_{\text{increasingly strong versions of FM}} \longrightarrow \mathcal{T}_{\mathsf{color}_{<\omega}} \overset{\text{unless size } = 2}{\overset{\longleftarrow}{\longrightarrow}} \mathcal{T}_{\mathsf{LN}} \sqcup \mathcal{T}_{\mathsf{sep}}$$

$$\mathcal{T}_{\mathsf{color}_{<\omega}} \upharpoonleft\!\downharpoonright \mathcal{T}_{\mathsf{color}_{\leq 3}}$$

can be reversed. However, these can be.

Interpretations in the wild

Example (Feldman–Moore) $\mathcal{T}_{\mathbb{F}_\omega}$ = "transitive $\mathbb{F}_\omega$-actions" $\to \mathcal{T}_{\mathsf{LN}} \sqcup \mathcal{T}_{\mathsf{sep}}$.
In fact, $\mathcal{T}_{\mathbb{Z}_2^{*\omega}} \to \mathcal{T}_{\mathsf{LN}} \sqcup \mathcal{T}_{\mathsf{sep}}$.
In fact, $\mathcal{T}_{\mathsf{color}_2}$ = "ω-colorings of complete graph" $\to \mathcal{T}_{\mathsf{LN}} \sqcup \mathcal{T}_{\mathsf{sep}}$.

Example (Kechris–Miller)
$\mathcal{T}_{\mathsf{color}_{<\omega}}$ = "ω-colorings of complete $< \omega$-hypergraph" $\to \mathcal{T}_{\mathsf{LN}} \sqcup \mathcal{T}_{\mathsf{sep}}$.

Example (Slaman–Steel) $\mathcal{T}_{\mathsf{marker}}$ = "$\bigcap\!\!\downarrow_n A_n = \varnothing$, $A_n \neq \varnothing$" $\to \mathcal{T}_{\mathsf{sep}} \sqcup \mathcal{T}_{\mathsf{inf}}$.

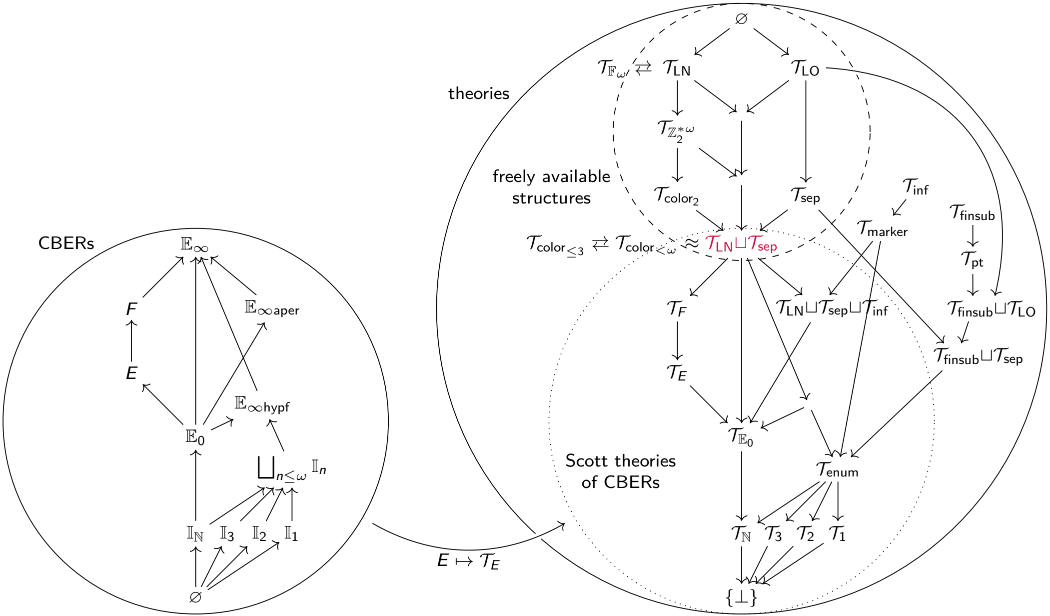
Theorem (Banerjee-C.) *None of the interpretabilities*

$$\underbrace{\mathcal{T}_{\mathsf{LN}} \rightleftarrows \mathcal{T}_{\mathbb{F}_\omega} \longrightarrow \mathcal{T}_{\mathbb{Z}_2^{*\omega}} \longrightarrow \mathcal{T}_{\mathsf{color}_2}}_{\textit{increasingly strong versions of FM}} \longrightarrow \mathcal{T}_{\mathsf{color}_{<\omega}} \overset{\textit{unless size}=2}{\overset{\leftarrow}{\longrightarrow}} \mathcal{T}_{\mathsf{LN}} \sqcup \mathcal{T}_{\mathsf{sep}}$$

$$\mathcal{T}_{\mathsf{color}_{<\omega}} \uparrow\!\downarrow \mathcal{T}_{\mathsf{color}_{\leq 3}}$$

can be reversed. However, these can be.

Proofs of $\not\leftarrow$: e.g., $(\mathbb{Z}, (-) + n)_{n \in \mathbb{Z}} \models \mathcal{T}_{\mathsf{LN}}$, but has nontrivial automorphisms.



CBERs
$\mathbb{E}_\infty$
F
$\mathbb{E}_{\infty\mathsf{aper}}$
E
$\mathbb{E}_{\infty\mathsf{hypf}}$
$\mathbb{E}_0$
$\bigsqcup_{n\le\omega}\mathbb{I}_n$
$\mathbb{I}_\mathbb{N}$ $\mathbb{I}_3$ $\mathbb{I}_2$ $\mathbb{I}_1$
$\varnothing$
$E \mapsto \mathcal{T}_E$
theories
freely available structures
Scott theories of CBERs
$\varnothing$
$\mathcal{T}_{\mathbb{F}_\omega} \rightleftarrows \mathcal{T}_\mathsf{LN}$
$\mathcal{T}_\mathsf{LO}$
$\mathcal{T}_{\mathbb{Z}_2^{*\omega}}$
$\mathcal{T}_{\mathsf{color}_2}$
$\mathcal{T}_\mathsf{sep}$
$\mathcal{T}_\mathsf{inf}$
$\mathcal{T}_\mathsf{finsub}$
$\mathcal{T}_\mathsf{marker}$
$\mathcal{T}_{\mathsf{color}_{\le 3}} \rightleftarrows \mathcal{T}_{\mathsf{color}_{<\omega}} \approx \mathcal{T}_\mathsf{LN} \sqcup \mathcal{T}_\mathsf{sep}$
$\mathcal{T}_\mathsf{pt}$
$\mathcal{T}_F$
$\mathcal{T}_\mathsf{LN} \sqcup \mathcal{T}_\mathsf{sep} \sqcup \mathcal{T}_\mathsf{inf}$
$\mathcal{T}_\mathsf{finsub} \sqcup \mathcal{T}_\mathsf{LO}$
$\mathcal{T}_\mathsf{finsub} \sqcup \mathcal{T}_\mathsf{sep}$
$\mathcal{T}_E$
$\mathcal{T}_{\mathbb{E}_0}$
$\mathcal{T}_\mathsf{enum}$
$\mathcal{T}_\mathbb{N}$ $\mathcal{T}_3$ $\mathcal{T}_2$ $\mathcal{T}_1$
$\{\bot\}$