Indices of O-regular variation and the Borel map in Carleman-Roumieu ultraholomorphic classes in sectors

Javier Sanz
Departamento de Álgebra, Análisis Matemático, Geometría y Topología
Universidad de Valladolid

Complex Analysis Seminar, Universität Wien, Austria
July 2nd, 2019

Ultradifferentiable classes by J. Hadamard

$\mathbb{N}_0 = \{0, 1, 2, ...\}$, $\mathbb{N} = \{1, 2, 3...\}$.

Definition (J. Hadamard (1912))

Given a sequence of positive real numbers $\mathbb{M} = (M_n)_{n\in\mathbb{N}_0}$ and an interval I in $\mathbb{R}$, the (Roumieu) ultradifferentiable class $\mathcal{C}_{\{M_n\}}(I)$ (or $\mathcal{C}_{\{\mathbb{M}\}}(I)$) consists of the complex smooth functions f defined in I for which there exist $C = C(f) > 0$ and $A = A(f) > 0$ such that

$$|f^{(n)}(x)| \le CA^n M_n, \qquad n \in \mathbb{N}_0,\ x \in I.$$

In this case, we say $f \in \mathcal{C}_{\{\mathbb{M}\},A}(I)$, and $\mathcal{C}_{\{\mathbb{M}\}}(I) = \cup_{A>0}\mathcal{C}_{\{\mathbb{M}\},A}(I)$.

Suppose $0 \in I$. The Borel map $\mathcal{B}$ is given by $\mathcal{B}(f) = (f^{(n)}(0))_{n\in\mathbb{N}_0}$.

The Borel map. Quasianalytic classes

$\mathcal{C}_{\{\mathbb{M}\}}(I)$ is said to be quasianalytic if whenever $f \in \mathcal{C}_{\{\mathbb{M}\}}(I)$ and $f^{(n)}(0) = 0$ for all n, then $f \equiv 0$.
We will always assume that $M_0 = 1$, and that $\mathbb{M}$ is logarithmically convex (lc), i.e. $M_n^2 \leq M_{n-1}M_{n+1}$, $n \geq 1$ (in other words, the sequence of quotients, $\boldsymbol{m} = (m_n := M_{n+1}/M_n)_{n\in\mathbb{N}_0}$, is nondecreasing).

Theorem (Denjoy-Carleman)

$\mathcal{C}_{\{\mathbb{M}\}}(I)$ *is quasianalytic* $\Leftrightarrow \sum_{n=0}^{\infty} \dfrac{1}{m_n} = \infty$.

Surjectivity of the Borel map

If $f \in \mathcal{C}_{\{\mathbb{M}\},A}(I)$, then $\mathcal{B}(f) = (f^{(n)}(0))_{n\in\mathbb{N}_0} \in \Lambda_{\{\mathbb{M}\},A}$, where
$\Lambda_{\{\mathbb{M}\},A} := \{(a_n)_{n\in\mathbb{N}_0} : |a_n| \le CA^n M_n$ for some $C\}$; $\Lambda_{\{\mathbb{M}\}} := \cup_{A>0}\Lambda_{\{\mathbb{M}\},A}$.

Theorem (H.-J. Petzsche (1988))

The Borel map $\mathcal{B} : \mathcal{C}_{\{\mathbb{M}\}}(I) \to \Lambda_{\{\mathbb{M}\}}$ is surjective if and only if there exists $C > 0$ such that

$$\sum_{q=n}^{\infty} \frac{1}{m_q} \le C\frac{n}{m_n}, \quad n \in \mathbb{N}. \text{ (Strong non-quasianalyticity condition)}$$

In this case, there exists $c > 0$ such that for every $A > 0$ there exists a right inverse for $\mathcal{B}$, $T_A : \Lambda_{\{\mathbb{M}\},A} \to \mathcal{C}_{\{\mathbb{M}\},cA}(I)$.

Holomorphic systems of ODEs in the complex domain

Theorem

Let the n-fold vector function $f(z, y)$ be holomorphic in a domain D of $\mathbb{C} \times \mathbb{C}^n$, and $(z_0, y_0) \in D$. Then, the Cauchy problem

$$y' = f(z, y), \quad y(z_0) = y_0$$

has a unique holomorphic solution at z_0.

Summability theory for divergent power series

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Objective: to build up sectorial analytic solutions to such equations, departing from the formal, but with Gevrey growth (E. Maillet (1903)), power series solutions which, in fact, will asymptotically represent the former.

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Tools for Gevrey (multi)summability (J.P. Ramis, J. Écalle, B. Malgrange, W. Balser):

- Gevrey asymptotics
- (Formal and analytic) analogues of the Laplace and Borel transforms

Theorem (B.L.J. Braaksma (1992))

Every formal power series solution to a nonlinear meromorphic system of ordinary differential equations at an irregular singular point is multisummable.

Difference equations

Linear difference equations may have formal power series solutions that are not Gevrey multisummable. Consider, for example, the inhomogeneous equation

$$y(z+1) - \frac{a}{z} y(z) = \frac{1}{z}$$

where $a \neq 0$. It has a unique formal power series solution $\hat{f} = \sum_{n=1}^{\infty} a_n z^{-n}$ with the property that

$$|a_n| \leq C A^n \frac{n!}{(\log n)^n}$$

for every $n \geq 2$ and suitable $A, C > 0$.
These equations are said to have a "level 1^+", and in this case (Gevrey) multisummability fails. However, G.K. Immink (2001, 2008, 2011) has introduced a non-standard multisummability procedure for this specific situation.
Aim: Give a unified approach for (multi)summability, what requires the study of the injectivity and surjectivity of the (asymptotic) Borel map.

Weight sequences, examples

We always assume that $\mathbb{M}$ is (lc) and moreover $\lim_{n\to\infty} m_n = \infty$: we say $\mathbb{M}$ is a weight sequence and we write $\mathbb{M} \in \mathcal{LC}$ for short.

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Examples:

- $\mathbb{M} = (\prod_{k=0}^{n} \log^{\beta}(e+k))_{n\in\mathbb{N}_0}$, $\beta > 0$, $m_n = \log^{\beta}(e+n+1)$.
- $\mathbb{M}_\alpha = (n!^\alpha)_{n\in\mathbb{N}_0}$, Gevrey sequence of order $\alpha > 0$, $m_n = (n+1)^\alpha$.
- $\mathbb{M}_{\alpha,\beta} = \left(n!^\alpha \prod_{m=0}^{n} \log^{\beta}(e+m)\right)_{n\in\mathbb{N}_0}$, $\alpha > 0$, $\beta \in \mathbb{R}$, $m_n = (n+1)^\alpha \log^{\beta}(e+n+1)$. ($\mathbb{M}_{1,-1}$ appears for difference equations.)
- For $q > 1$, $\mathbb{M} = (q^{n^2})_{n\in\mathbb{N}_0}$, $q-$Gevrey sequence, $m_n = q^{2n+1}$. (q-difference equations.)

Sectors and sectorial regions

$\mathcal{R}$ will denote the Riemann surface of the logarithm.

Let $d \in \mathbb{R}$, $\gamma, r > 0$:
A bounded sector (with vertex at 0) of radius r, bisected by direction d and with opening $\pi\gamma$ is

$$S(d,\gamma,r) := \{z \in \mathcal{R};\ |\arg(z) - d| < \pi\gamma/2,\ |z| < r\}$$

For unbounded sectors, we write

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A sectorial region, $G(d,\gamma)$, is an open connected set in $\mathcal{R}$ such that $G(d,\gamma) \subset S(d,\gamma)$, and for every $\beta \in (0,\gamma)$ there exists $r = r(\beta) > 0$ with

$$\overline{S(d,\beta,r)} \subset G(d,\gamma).$$

In this case we say that $T := S(d,\beta,r)$ is a bounded proper subsector of G.
In particular, sectors are sectorial regions.
If $d = 0$, we write $S_\gamma := S(0,\gamma)$, $G_\gamma := G(0,\gamma)$.

Ultraholomorphic (Roumieu-Carleman) classes

Given $\mathbb{M}$ and a sector S, we consider

$$\mathcal{A}_{\mathbb{M}}(S)=\left\{f\in\mathcal{H}(S):\exists A>0 \text{ s.t. } \sup_{z\in S,n\in\mathbb{N}_0}\frac{|f^{(n)}(z)|}{A^n n!M_n}<\infty\right\}.$$

For $f\in\mathcal{A}_{\mathbb{M}}(S)$ and for every $n\in\mathbb{N}_0$, there exists

$$f^{(n)}(0):=\lim_{z\to 0,z\in S}f^{(n)}(z),$$

and the formal (generally divergent) series $\widehat{f}=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}z^n$ satisfies $|f^{(n)}(0)|\le CA^n n!M_n$ for some $C,A>0$, and we write

$$\widehat{f}\in\mathbb{C}[[z]]_{\mathbb{M}}.$$

Asymptotics

$f : G \to \mathbb{C}$ (holomorphic in a sectorial region G) admits the series $\hat{f} = \sum_{n=0}^{\infty} a_n z^n$ as its $\mathbb{M}$-asymptotic expansion at 0, denoted $f \sim_{\mathbb{M}} \hat{f}$, if for every bounded proper subsector T of G there exist $C_T, B_T > 0$ such that for every $z \in T$ and every $n \in \mathbb{N}_0$, we have

$$\Big|f(z) - \sum_{k=0}^{n-1} a_k z^k\Big| \le C_T B_T^n M_n |z|^n. \qquad [f \in \widetilde{\mathcal{A}}_{\mathbb{M}}(G)]$$

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f admits the series $\hat{f} = \sum_{n=0}^{\infty} a_n z^n$ as its $\mathbb{M}$-uniform asymptotic expansion at 0 if there exist $C, B > 0$ such that for every $z \in G$ and every $n \in \mathbb{N}_0$, we have

$$\Big|f(z) - \sum_{k=0}^{n-1} a_k z^k\Big| \le C B^n M_n |z|^n. \qquad [f \in \widetilde{\mathcal{A}}^u_{\mathbb{M}}(G)]$$

For any sector S and any bounded proper subsector T of S,

$$\mathcal{A}_{\mathbb{M}}(S) \subset \widetilde{\mathcal{A}}^u_{\mathbb{M}}(S) \subset \widetilde{\mathcal{A}}_{\mathbb{M}}(S) \subset \mathcal{A}_{\mathbb{M}}(T),$$

and for f in any of these spaces, $a_n = f^{(n)}(0)/n!$ for every n.

The asymptotic Borel map

We consider the asymptotic Borel map (homomorphism of algebras)

$$\begin{aligned}\widetilde{\mathcal{B}}:\mathcal{A}_{\mathbb{M}}(S),\ \widetilde{\mathcal{A}}^{u}_{\mathbb{M}}(G),\ \widetilde{\mathcal{A}}_{\mathbb{M}}(G) &\longrightarrow \mathbb{C}[[z]]_{\mathbb{M}}\\ f &\mapsto \hat{f}=\sum_{n=0}^{\infty}a_nz^n.\end{aligned}$$

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A function f in any of these classes is said to be flat if $f\sim_{\mathbb{M}}\hat{0}$, the null formal power series.
The class is said to be quasianalytic if it does not contain nontrivial flat functions.

Injectivity intervals

By a simple rotation, the injectivity and surjectivity of the Borel map do not depend on the bisecting direction.

We define

$$I_{\mathbb{M}} := \{\gamma > 0 : \quad \widetilde{\mathcal{B}} : \mathcal{A}_{\mathbb{M}}(S_\gamma) \longrightarrow \mathbb{C}[[z]]_{\mathbb{M}} \text{ is injective}\},$$

$$\widetilde{I}^u_{\mathbb{M}} := \{\gamma > 0 : \quad \widetilde{\mathcal{B}} : \widetilde{\mathcal{A}}^u_{\mathbb{M}}(S_\gamma) \longrightarrow \mathbb{C}[[z]]_{\mathbb{M}} \text{ is injective}\},$$

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By the identity principle, $I_{\mathbb{M}}$, $\widetilde{I}^u_{\mathbb{M}}$ and $\widetilde{I}_{\mathbb{M}}$ are either empty or unbounded intervals contained in $(0,\infty)$. Moreover, since

$$\mathcal{A}_{\mathbb{M}}(S_\gamma) \subseteq \widetilde{\mathcal{A}}^u_{\mathbb{M}}(S_\gamma) \subseteq \widetilde{\mathcal{A}}_{\mathbb{M}}(S_\gamma),$$

we have

$$I_{\mathbb{M}} \supseteq \widetilde{I}^u_{\mathbb{M}} \supseteq \widetilde{I}_{\mathbb{M}}.$$

Classical injectivity results

S. Mandelbrojt, *Séries adhérentes, régularisation des suites, applications*, Collection de monographies sur la théorie des fonctions, Gauthier-Villars, Paris, 1952.

Theorem

Let $\mathbb{M} \in \mathcal{LC}$, *then* $\widetilde{I}^u_{\mathbb{M}} = \{\gamma > 0 : \sum_{n=0}^{\infty} \left(\frac{1}{m_n}\right)^{1/\gamma}$ *diverges*$\}$.

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B. Rodríguez Salinas, Functions with null moments (Spanish), Rev. Acad. Ciencias, 49 (1955), 331–368.
B. I. Korenbljum, Conditions of nontriviality of certain classes of functions analytic in a sector, and problems of quasianalyticity, Soviet Math. Dokl. 7 (1966), 232–236.

Theorem

Let $\mathbb{M} \in \mathcal{LC}$, *then* $I_{\mathbb{M}} = \{\gamma > 0 : \sum_{n=0}^{\infty} \left(\frac{1}{(n+1)m_n}\right)^{1/(\gamma+1)}$ *diverges*$\}$.

Optimal opening for quasianalyticity

J. Jiménez-Garrido, J. S., Strongly regular sequences and proximate orders. J. Math. Anal. Appl. 438 (2016), no. 2, 920–945.

For $\mathbb{M} \in \mathcal{LC}$, the value that tells apart quasianalyticity from non-quasianalyticity in sectorial regions G_γ is the inverse of the convergence exponent of $\boldsymbol{m}$, i.e.,

$$\omega(\mathbb{M}) := \liminf_{n\to\infty} \frac{\log(m_n)}{\log(n)} \in [0,\infty].$$

If $\omega(\mathbb{M}) = 0$, $I_{\mathbb{M}} = \widetilde{I}^u_{\mathbb{M}} = \widetilde{I}_{\mathbb{M}} = (0,\infty)$. If $\omega(\mathbb{M}) = \infty$, $I_{\mathbb{M}} = \widetilde{I}^u_{\mathbb{M}} = \widetilde{I}_{\mathbb{M}} = \emptyset$.

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	$\sum_{p=0}^{\infty}\sigma_p := \sum_{p=0}^{\infty}\left(\frac{1}{(p+1)m_p}\right)^{\frac{1}{\omega(\mathbb{M})+1}} = \infty$; $\sum_{p=0}^{\infty}\mu_p := \sum_{p=0}^{\infty}\left(\frac{1}{m_p}\right)^{\frac{1}{\omega(\mathbb{M})}} = \infty$	$\sum_{p=0}^{\infty}\sigma_p = \infty$; $\sum_{p=0}^{\infty}\mu_p < \infty$	$\sum_{p=0}^{\infty}\sigma_p < \infty$; $\sum_{p=0}^{\infty}\mu_p < \infty$
$I_{\mathbb{M}}$	$[\omega(\mathbb{M}),\infty)$	$[\omega(\mathbb{M}),\infty)$	$(\omega(\mathbb{M}),\infty)$
$\widetilde{I}^u_{\mathbb{M}}$	$[\omega(\mathbb{M}),\infty)$	$(\omega(\mathbb{M}),\infty)$	$(\omega(\mathbb{M}),\infty)$
$\widetilde{I}_{\mathbb{M}}$	$(\omega(\mathbb{M}),\infty)$ or $[\omega(\mathbb{M}),\infty)$	$(\omega(\mathbb{M}),\infty)$	$(\omega(\mathbb{M}),\infty)$

Auxiliary functions associated with $\mathbb{M}$ and flatness

Flatness for $f \in \widetilde{\mathcal{A}}_{\mathbb{M}}(G)$ amounts to: For every bounded proper subsector T of G there exist $C_T, B_T > 0$ with

$$|f(z)| \leq \inf_{n\in\mathbb{N}_0} C_T B_T^n M_n |z|^n = C_T h_{\mathbb{M}}(B_T|z|) = C_T e^{-\omega_{\mathbb{M}}(1/(B_T|z|))}, \qquad z \in T,$$

where, for $t > 0$,

$$h_{\mathbb{M}}(t) := \inf_{n\geq 0} M_n t^n, \ t > 0; \quad \omega_{\mathbb{M}}(t) := \sup_{n\geq 0} \log \frac{t^n}{M_n}.$$

Proximate orders

Idea: For $|z|=r$, instead of comparing $\log|f(z)|$ to r^k (functions of finite exponential order), compare to $r^{\rho(r)}$ for some suitably chosen function $\rho(r)$ in $(0,\infty)$, so getting a refined scale of growth.

Definition (E. Lindelöf, G. Valiron)

We say $\rho(r):(0,\infty)\to\mathbb{R}$ is a proximate order if the following hold:

(1) ρ is continuous and piecewise continuously differentiable,

(2) $\rho(r)\geq 0$ for every $r>0$,

(3) $\lim_{r\to\infty}\rho(r)=\rho<\infty$,

(4) $\lim_{r\to\infty} r\rho'(r)\log(r)=0$.

In case $\lim_{r\to\infty}\rho(r)\in(0,\infty)$, we say $\rho(r)$ is a nonzero proximate order.

Example: $\rho_{\alpha,\beta}(t)=\dfrac{1}{\alpha}-\dfrac{\beta}{\alpha}\dfrac{\log(\log(t))}{\log(t)}$, $\alpha>0$, $\beta\in\mathbb{R}$.

Maergoiz's theorem

L. S. Maergoiz, Indicator diagram and generalized Borel-Laplace transforms for entire functions of a given proximate order, St. Petersburg Math. J. 12 (2001), no. 2, 191–232.

Theorem

Let $\rho(r)$ be a nonzero proximate order with $\rho(r) \to \rho > 0$ as $r \to \infty$. For every $\gamma > 0$ there exists an analytic function $V(z)$ in S_γ such that:

(1) $\displaystyle\lim_{r\to\infty} \frac{V(zr)}{V(r)} = z^\rho$ *uniformly in the compact sets of S_γ (regular variation).*

(2) $\overline{V(z)} = V(\overline{z})$ *for every $z \in S_\gamma$.*

(3) $V(r)$ *is positive in $(0,\infty)$.*

(4) $\displaystyle\lim_{r\to\infty} \frac{V(r)}{r^{\rho(r)}} = 1.$

We say V is a function of Maergoiz for $\rho(r)$ in S_γ.

Complete solution for injectivity

J. Jiménez-Garrido, PhD Dissertation, University of Valladolid, 2018.

J. Jiménez-Garrido, J. S., G. Schindl, Injectivity and surjectivity of the asymptotic Borel map in Carleman ultraholomorphic classes, J. Math. Anal. Appl. 469 (2019), 136–168.

Theorem (general Watson's lemma)

Suppose $\mathbb{M} \in \mathcal{LC}$ *and* $\omega(\mathbb{M}) \in (0, \infty)$. *Then,* $\widetilde{\mathcal{A}}_{\mathbb{M}}(S_{\omega(\mathbb{M})})$ *is not quasianalytic.*

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	$\sum_{p=0}^{\infty} \sigma_p = \infty$, $\sum_{p=0}^{\infty} \mu_p = \infty$	$\sum_{p=0}^{\infty} \sigma_p = \infty$, $\sum_{p=0}^{\infty} \mu_p < \infty$	$\sum_{p=0}^{\infty} \sigma_p < \infty$, $\sum_{p=0}^{\infty} \mu_p < \infty$
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Definition of $\mathbb{M}$-summability

A. Lastra, S. Malek, J. S., Summability in general Carleman ultraholomorphic classes, J. Math. Anal. Appl. 430 (2015), 1175–1206.

Let $\mathbb{M} \in \mathcal{LC}$, $d \in \mathbb{R}$. We say $\hat{f} = \sum_{n\geq 0} \dfrac{f_n}{n!} z^n$ is $\mathbb{M}$-summable in direction d if there exist a sectorial region $G = G(d, \gamma)$, with $\gamma > \omega(\mathbb{M})$, and $f \in \widetilde{\mathcal{A}}_{\mathbb{M}}(G)$ such that $f \sim_{\mathbb{M}} \hat{f}$.

In this case, f is unique and will be called the $\mathbb{M}$-sum of $\hat{f}$ in direction d, denoted by $\mathcal{S}_{\mathbb{M},d}\hat{f}$.

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For the explicit construction of the sum of an $\mathbb{M}-$summable series in a direction d we follow the ideas of moment summability methods.
W. Balser, *Formal power series and linear systems of meromorphic ordinary differential equations*, Springer, Berlin, 2000.

Admissibility of a proximate order and kernels

We say $\mathbb{M}$ admits a nonzero proximate order if there exists a nonzero proximate order $\rho(r)$ and constants $A, B > 0$ with

$$A \leq \frac{\omega_{\mathbb{M}}(r)}{r^{\rho(r)}} \leq B, \quad r \text{ large enough.}$$

Example: $\omega_{\mathbb{M}_{\alpha,\beta}}(r)$ is comparable to $r^{1/\alpha}\log^{-\beta/\alpha}(r)$, so $\mathbb{M}_{\alpha,\beta}$ admits the proximate order

$$\rho_{\alpha,\beta}(r) = \frac{1}{\alpha} - \frac{\beta}{\alpha}\frac{\log(\log(r))}{\log(r)}.$$

Theorem

If $\mathbb{M} \in \mathcal{LC}$ admits a nonzero proximate order, there exist kernels of $\mathbb{M}-$summability and associated Laplace- and Borel-like transforms, both formal and analytic, that allow for the reconstruction of the $\mathbb{M}-$sum of any $\mathbb{M}-$summable series in a direction.

Surjectivity intervals

We define now

$$\begin{aligned} S_{\mathbb{M}} &:= \{\gamma > 0; \quad \widetilde{\mathcal{B}} : \mathcal{A}_{\mathbb{M}}(S_\gamma) \longrightarrow \mathbb{C}[[z]]_{\mathbb{M}} \text{ is surjective}\}, \\ \widetilde{S}^u_{\mathbb{M}} &:= \{\gamma > 0; \quad \widetilde{\mathcal{B}} : \widetilde{\mathcal{A}}^u_{\mathbb{M}}(S_\gamma) \longrightarrow \mathbb{C}[[z]]_{\mathbb{M}} \text{ is surjective}\}, \\ \widetilde{S}_{\mathbb{M}} &:= \{\gamma > 0; \quad \widetilde{\mathcal{B}} : \widetilde{\mathcal{A}}_{\mathbb{M}}(S_\gamma) \longrightarrow \mathbb{C}[[z]]_{\mathbb{M}} \text{ is surjective}\}. \end{aligned}$$

Surjectivity intervals

We define now

$$\begin{aligned} S_{\mathbb{M}} :=& \{\gamma > 0; \quad \widetilde{\mathcal{B}} : \mathcal{A}_{\mathbb{M}}(S_\gamma) \longrightarrow \mathbb{C}[[z]]_{\mathbb{M}} \text{ is surjective}\}, \\ \widetilde{S}^u_{\mathbb{M}} :=& \{\gamma > 0; \quad \widetilde{\mathcal{B}} : \widetilde{\mathcal{A}}^u_{\mathbb{M}}(S_\gamma) \longrightarrow \mathbb{C}[[z]]_{\mathbb{M}} \text{ is surjective}\}, \\ \widetilde{S}_{\mathbb{M}} :=& \{\gamma > 0; \quad \widetilde{\mathcal{B}} : \widetilde{\mathcal{A}}_{\mathbb{M}}(S_\gamma) \longrightarrow \mathbb{C}[[z]]_{\mathbb{M}} \text{ is surjective}\}. \end{aligned}$$

We deduce that $S_{\mathbb{M}}$, $\widetilde{S}^u_{\mathbb{M}}$ and $\widetilde{S}_{\mathbb{M}}$ are either empty or left-open intervals having 0 as endpoint.
Since

$$\mathcal{A}_{\mathbb{M}}(S_\gamma) \subseteq \widetilde{\mathcal{A}}^u_{\mathbb{M}}(S_\gamma) \subseteq \widetilde{\mathcal{A}}_{\mathbb{M}}(S_\gamma),$$

we see that

$$S_{\mathbb{M}} \subseteq \widetilde{S}^u_{\mathbb{M}} \subseteq \widetilde{S}_{\mathbb{M}}.$$

Conditions for sequences

Let $\mathbb{M} = (M_n)_{n\in\mathbb{N}_0}$ be a sequence of positive numbers with $M_0 = 1$.

- $\mathbb{M}$ has moderate growth (mg) if there exists a constant $A > 0$ such that

$$M_{n+p} \leq A^{n+p} M_n M_p, \quad n, p \in \mathbb{N}_0.$$

- $\mathbb{M}$ is strongly non-quasianalytic (snq) if there exists $B > 0$ such that

$$\sum_{k\geq n} \frac{M_k}{(k+1)M_{k+1}} \leq B \frac{M_n}{M_{n+1}}, \quad n \in \mathbb{N}_0.$$

$\mathbb{M}$ is strongly regular if it is (lc), (mg) and (snq).

Example: $\mathbb{M}_{\alpha,\beta}$ is strongly regular.

Borel-Ritt-Gevrey theorem. Thilliez's index

J. P. Ramis, Dévissage Gevrey, Asterisque 59–60 (1978), 173–204.

Theorem (Borel–Ritt–Gevrey, J. P. Ramis)

For the Gevrey sequence $\mathbb{M}_\alpha$, $\alpha > 0$, $(0, \infty)$ *is the disjoint union of the intervals of injectivity and surjectivity for the three classes considered.*

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V. Thilliez introduces a growth index for strongly regular sequences.

Definition (V. Thilliez (2003))

Let $\mathbb{M} = (M_p)_{p\in\mathbb{N}_0}$ be a strongly regular sequence and $\gamma > 0$.

$\mathbb{M}$ satisfies property (P_γ) if there exist a sequence of real numbers $m' = (m'_p)_{p\in\mathbb{N}_0}$ and a constant $a \geq 1$ such that:

(i) $a^{-1}m_p \leq m'_p \leq a m_p$, $p \in \mathbb{N}$, and (ii) $\left((p+1)^{-\gamma} m'_p\right)_{p\in\mathbb{N}_0}$ is increasing.

Then,

$$\gamma(\mathbb{M}) := \sup\{\gamma > 0 : (P_\gamma) \text{ is satisfied}\}.$$

One has $\gamma(\mathbb{M}) \in (0, \infty)$.

Thilliez's result. Idea of proof

Theorem (V. Thilliez (2003))

Let $\mathbb{M}$ be a strongly regular sequence and $0 < \gamma < \gamma(\mathbb{M})$. Then $\widetilde{\mathcal{B}} : \mathcal{A}_{\mathbb{M}}(S_\gamma) \longrightarrow \mathbb{C}[[z]]_{\mathbb{M}}$ is surjective. Moreover, there exists $c \geq 1$, depending only on $\mathbb{M}$ and γ, such that for every $A > 0$ there exists a right inverse for $\widetilde{\mathcal{B}}$, $U_{\mathbb{M},A,\gamma} : \mathbb{C}[[z]]_{\mathbb{M},A} \to \mathcal{A}_{\mathbb{M},cA}(S_\gamma)$.

Steps of the proof:

(i) Given $0 < \delta < \gamma(\mathbb{M})$, V. Thilliez constructed "optimal" flat functions $G : S_\delta \to \mathbb{C}$ such that: there exist $k_1, k_2, k_3 > 0$ with

$$k_1 h_{\mathbb{M}}(k_2|z|) \leq |G(z)| \leq h_{\mathbb{M}}(k_3|z|), \qquad z \in S_\delta.$$

(ii) Use of two different versions of Whitney extension results from the ultradifferentiable setting (J. Bruna; H.-J. Petzsche; J. Bonet, R. W. Braun, R. Meise and B. A. Taylor, and J. Chaumat and A.-M. Chollet (1980-1994)).

Open problems and first answer

Many questions arose:

- Thilliez indicates that, due to Petzsche's result, (snq) is necessary for surjectivity in this case, but (mg) seems to be a technical condition.
- Since (optimal) flat functions are constructed in the proof, the classes $\mathcal{A}_{\mathbb{M}}(S_\gamma)$ are nonquasianalytic. Are injectivity and surjectivity always incompatible?

Theorem (J. Jiménez-Garrido, J. S., G. Schindl)

Let $\mathbb{M} \in \mathcal{LC}$. Then, the Borel map is never bijective in any of the classes considered and in any sector.

- Will the splitting of $(0, \infty)$ in Borel–Ritt–Gevrey theorem persist for any strongly regular sequence? And in other cases?
- Which is the precise relation between $\gamma(\mathbb{M})$ and $\omega(\mathbb{M})$? In the classical examples they always coincide, but only $\gamma(\mathbb{M}) \leq \omega(\mathbb{M})$ is clear.
- No information is given about the optimality of $\gamma(\mathbb{M})$ in the previous result of Thilliez.
- Is the tool of proximate orders relevant also for surjectivity?

O-regular variation for sequences

The theory of O-regular variation of functions was started by J. Karamata in the 1930's and mainly developed by authors of the Serbian school. It studies different indices and orders measuring the growth of a positive function in an unbounded interval.
The extension of the notion of O-regular variation for sequences was stated by S. Aljančić (1981) and studied by D. Djurčić and V. Božin (1997); however, they did not study in detail indices and orders for sequences.

Definition

A sequence $\boldsymbol{m} = (m_p)_{p\in\mathbb{N}}$ of positive numbers is said to be O-regularly varying if

$$\limsup_{n\to\infty} \frac{m_{\lfloor \lambda n\rfloor}}{m_n} < \infty$$

for every $\lambda \in (0,\infty)$.

J. Jiménez-Garrido, J. S., G. Schindl, Indices of O-regular variation for weight functions and weight sequences, submitted, available at http://arxiv.org/abs/1806.01605.

$\omega(\mathbb{M})$ and $\gamma(\mathbb{M})$ are indices of O-regular variation

The sequence $\boldsymbol{m}$ is said to be almost increasing (resp. almost decreasing) if there exists some $K > 0$ such that $m_p \leq K m_q$ (resp. $m_p \geq K m_q$) for all $p, q \in \mathbb{N}$ with $p \leq q$.

Proposition (J. Jiménez-Garrido, J. S., G. Schindl)

We have that

$$\beta(\boldsymbol{m}) = \sup\{\beta \in \mathbb{R} : (m_n/n^\beta)_{n\in\mathbb{N}} \text{ is almost increasing}\} = \gamma(\mathbb{M}) \text{ (V. Thilliez)},$$

$$\mu(\boldsymbol{m}) = \liminf_{n\to\infty} \frac{\log(m_n)}{\log(n)} = \omega(\mathbb{M}),$$

$$\rho(\boldsymbol{m}) = \limsup_{n\to\infty} \frac{\log(m_n)}{\log(n)},$$

$$\alpha(\boldsymbol{m}) = \inf\{\alpha \in \mathbb{R}; (m_n/n^\alpha)_{n\in\mathbb{N}} \text{ is almost decreasing}\}.$$

Moreover, $-\infty \leq \gamma(\mathbb{M}) \leq \omega(\mathbb{M}) \leq \rho(\boldsymbol{m}) \leq \alpha(\boldsymbol{m}) \leq \infty$.
$\boldsymbol{m}$ *is O-regularly varying if and only if all its indices are finite.*

(snq) and (mg) expressed by indices

Proposition (S. Tikhonov (2004))

Let $\mathbb{M} \in \mathcal{LC}$, *then* $\mathbb{M}$ *is (snq) if and only if* $\gamma(\mathbb{M}) > 0$.

(snq) and (mg) expressed by indices

Proposition (S. Tikhonov (2004))

Let $\mathbb{M} \in \mathcal{LC}$, then $\mathbb{M}$ is (snq) if and only if $\gamma(\mathbb{M}) > 0$.

H.-J. Petzsche, D.Vogt, Almost analytic extension of ultradifferentiable functions and the boundary values of holomorphic functions, Math. Ann. 267 (1984), 17–35.
W. Matsumoto, Characterization of the separativity of ultradifferentiable classes, J. Math. Kyoto Univ. 24, no. 4 (1984), 667–678.

Corollary

Let $\mathbb{M} \in \mathcal{LC}$. $\mathbb{M}$ satisfies (mg) if and only if $\alpha(\boldsymbol{m}) < \infty$.
$\mathbb{M}$ is strongly regular if and only if all its indices are positive real numbers.

Indices of O-regular variation may be arbitrarily prescribed

In general, given positive real numbers $0 < \gamma \le \omega \le \rho \le \alpha$ there exists a strongly regular sequence $\mathbb{M}$ such that

$$\gamma(\mathbb{M}) = \gamma, \quad \omega(\mathbb{M}) = \omega, \quad \rho(\boldsymbol{m}) = \rho, \quad \alpha(\boldsymbol{m}) = \alpha.$$

In particular, there exist strongly regular sequences with arbitrarily prescribed distinct indices $\gamma(\mathbb{M}) < \omega(\mathbb{M})$.

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J. Jiménez-Garrido, J. S., G. Schindl, Log-convex sequences and nonzero proximate orders, J. Math. Anal. Appl. 448, no. 2 (2017), 1572–1599.

Proposition (J. Jiménez-Garrido, J. S., G. Schindl (2017))

Let $\mathbb{M} \in \mathcal{LC}$ admit a nonzero proximate order $\rho(t) \to \rho > 0$. Then, $\boldsymbol{m}$ is O-regularly varying and all its indices coincide with $1/\rho$. In particular, $\mathbb{M}$ is strongly regular.

We deduce that not every strongly regular sequence admits a nonzero proximate order.

Maximal length of the surjectivity interval

Theorem (J. Jiménez-Garrido, J. S., G. Schindl)

If $\mathbb{M}$ is strongly regular and the Borel map in $\widetilde{\mathcal{A}}^u_{\mathbb{M}}(S_\gamma)$ is surjective, then $\gamma \leq \gamma(\mathbb{M})$.

J. Schmets, M. Valdivia, Extension maps in ultradifferentiable and ultraholomorphic function spaces, Studia Math. 143 (3) (2000), 221–250.

A crucial ramification argument works because of (mg).

If $\mathbb{M}$ is strongly regular, and except for the critical opening $\pi\gamma(\mathbb{M})$, surjectivity amounts to the existence of optimal flat functions.

Whenever $\gamma(\mathbb{M}) < \omega(\mathbb{M})$, one has three different situations in S_γ:

- $\gamma < \gamma(\mathbb{M})$: there are optimal flat functions.
- $\gamma(\mathbb{M}) < \gamma < \omega(\mathbb{M})$: there are nontrivial flat functions but no optimal one.
- $\gamma > \omega(\mathbb{M})$: there are no nontrivial flat functions.

Surjectivity for sequences admitting nonzero proximate order

In case $\mathbb{M}$ admits a nonzero proximate order, we improve the results thanks to the existence of kernels of $\mathbb{M}-$summability; moreover, $\gamma(\mathbb{M}) = \omega(\mathbb{M})$.

<table>
<tr><td rowspan="2"></td><td rowspan="2">$\gamma(\mathbb{M}) \in \mathbb{Q}$</td><td colspan="3">$\gamma(\mathbb{M}) \in \mathbb{I}$</td></tr>
<tr><td>$\sum_{p=0}^{\infty} \mu_p = \infty$</td><td>$\sum_{p=0}^{\infty} \mu_p < \infty, \sum_{p=0}^{\infty} \sigma_p = \infty$</td><td>$\sum_{p=0}^{\infty} \sigma_p < \infty$</td></tr>
<tr><td>$S_{\mathbb{M}}$</td><td colspan="3">$(0, \gamma(\mathbb{M}))$</td><td rowspan="2">$(0, \gamma(\mathbb{M}))$ or $(0, \gamma(\mathbb{M})]$</td></tr>
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Table: Surjectivity intervals for weight sequences admitting a nonzero proximate order.

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$S_{\mathbb{M}}$	$(0,\gamma(\mathbb{M}))$			
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Table: Surjectivity intervals for weight sequences admitting a nonzero proximate order.

One always has $(0,\infty)=\widetilde{I}_{\mathbb{M}}\cup\widetilde{S}_{\mathbb{M}}$.
If $\sum_{p=0}^{\infty}\mu_p=\infty$, then $(0,\infty)=\widetilde{I}^u_{\mathbb{M}}\cup\widetilde{S}^u_{\mathbb{M}}=I_{\mathbb{M}}\cup S_{\mathbb{M}}$ (for example, Gevrey case).
But if $\gamma(\mathbb{M})\in\mathbb{Q}$ and $\sum_{p=0}^{\infty}\sigma_p<\infty$, then the splitting fails:

$$\widetilde{I}^u_{\mathbb{M}}\cup\widetilde{S}^u_{\mathbb{M}}=I_{\mathbb{M}}\cup S_{\mathbb{M}}=(0,\infty)\setminus\{\gamma(\mathbb{M})\}.$$

A result for sequences of fast growth

While (mg) restricts from above the growth of the weight sequence ($\exists A, \alpha > 0 : M_n \leq A^n n!^\alpha,\ \forall n$), the following result for sequences of fast growth is inspired by Theorem 5.6 in J. Schmets, M. Valdivia (2000).

Theorem (J. Jiménez-Garrido, J. S., G. Schindl)

Let $\mathbb{M}$ be a weight sequence. The following are equivalent:

(1) $\gamma(\mathbb{M}) = \infty$,

(2) *For every $\gamma > 0$ there exists an extension operator $U_{\mathbb{M},\gamma} : \mathbb{C}[[z]]_{\mathbb{M}} \to \mathcal{A}_{\mathbb{M}}(S_\gamma)$. In particular, $S_{\mathbb{M}} = (0, \infty)$.*

Remark: Schmets and Valdivia imposed a condition stronger than $\gamma(\mathbb{M}) = \infty$, and also the condition

$$\forall \varepsilon > 0,\ \exists k \in \mathbb{N} : \limsup_{n\to\infty} \frac{1}{m_{kn}} \left(\frac{M_{kn}}{M_n} \right)^{1/((k-1)n)} \leq \varepsilon, \qquad (\beta_2)$$

but the theory of O-regular variation shows that $\gamma(\mathbb{M}) = \infty$ implies (β_2).

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Example: The $q-$Gevrey sequence $\mathbb{M}_q$ verifies the conditions of this theorem.

The study of surjectivity for weight sequences $\mathbb{M}$ with $\gamma(\mathbb{M}) \in (0, \infty)$ and $\alpha(\boldsymbol{m}) = \infty$ is work in progress.

Some results with derivation closedness

Consider a weaker condition: $\mathbb{M}$ is derivation closed (dc) if there exists a constant $A > 0$ such that

$$M_{n+1} \leq A^{n+1} M_n, \quad n \in \mathbb{N}_0.$$

If (dc) is imposed, there is some information:

(1) $S_{\mathbb{M}} \subseteq (0, \lfloor \gamma(\mathbb{M}) \rfloor + 1) \cap (0, \omega(\mathbb{M})]$.

(2) A. Debrouwere (2019): If $\gamma(\mathbb{M}) > 1$, then $(0, 1] \subset S_{\mathbb{M}}$.

An alternative proof of V. Thilliez's result

Alternative idea: Use an optimal flat function G in order to obtain a kernel $e(z) := zG(1/z)$ such that

(1) $e(x) > 0$ for every $x > 0$.

(2) There exist $c, k > 0$ such that $|e(z)| \le ch_{\mathbb{M}}(k/|z|)$ for every $z \in S_\gamma$.

(3) The sequence of moments, $m_e(n) = \int_0^\infty t^{n-1} e(t)\, dt$, $n \in \mathbb{N}_0$, is well defined and it is equivalent to $\mathbb{M}$ (only (dc) is needed).

Given $\hat{f} \in \mathbb{C}[[z]]_{\mathbb{M}}$, $g := \hat{T}_e^- \hat{f} = \sum_{n\ge 0} \dfrac{a_n}{m_e(n)} z^n$ converges; we apply a truncated Laplace-like transform

$$(T_e^t g)(z) := \int_0^{R(\tau)} e(u/z) g(u) \frac{du}{u}, \quad R \text{ suitably small.}$$

Then, $T_e^t g \sim_{\mathbb{M}} \hat{f}$, and we get surjectivity.

A special case: optimal flat functions for $q-$Gevrey sequences

For $q>1$ consider $\mathbb{M}_q=(q^{n^2})_{n\in\mathbb{N}_0}$; all the indices are infinite, and (dc) holds.

One can check that

$$\exp(-\frac{1}{4\log(q)}\log^2(t))\leq h_{\mathbb{M}_q}(t)\leq q\exp(-\frac{1}{4\log(q)}\log^2(q^2t)),$$

and then the function $G:\mathcal{R}\to\mathbb{C}$ given by

$$G(z)=\exp(-\frac{1}{4\log(q)}\log^2(z))$$

provides (by restriction) an optimal flat function in S_γ for every $\gamma>0$.
This shows that the failure of (mg) does not exclude the existence of optimal flat functions.

Thank you very much for your attention!