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Faculty of Mathematics



Mastering Math and Mind

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Preamble

This course contains basic learning and self-management techniques that are essential to succeed in the study of Mathematics at University level.

Here is an introductory video:

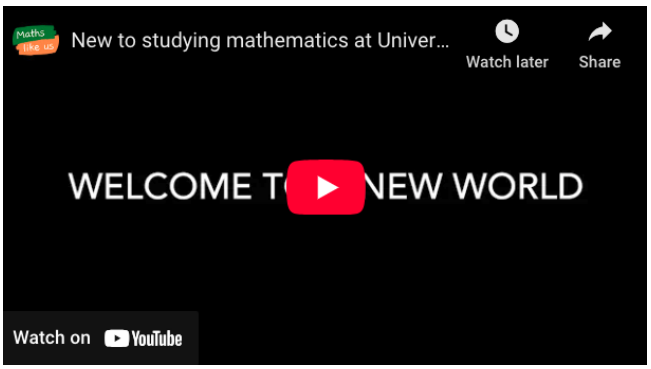


Figure 1: Introductory video to the course at https://www.youtube.com/watch?v=VFsMINQ72nU

This course is an initiative of Diversity-Equity-Inclusion at the Faculty of Mathematics of the University of Vienna.

Special thanks to Kata Sebök (researcher in didactics of mathematis) and Roland Steinbauer (researcher in mathematics) for all their ideas, insights and discussions.

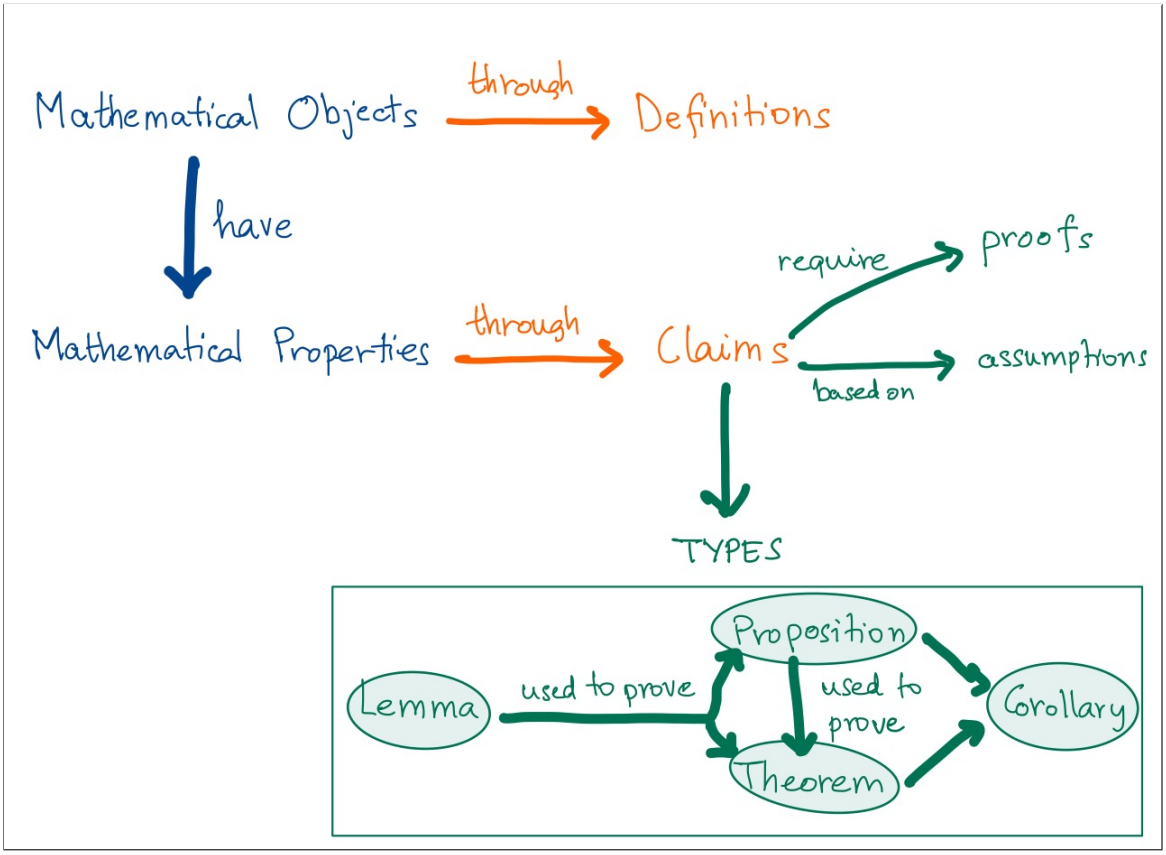
AI was used to polish parts of the text.

Part I

Math

Chapter 1

Elements of mathematics



1.1 Definitions: moving from vague language to precise mathematical formulation

Moving from vague language and handwaving arguments to precise mathematical formulation is not easy. The first and most important step towards it are definitions of mathematical objects or properties.

Definitions link common language with precise mathematical formulations.

For example, look at the definition of an even number:
Definition 1: An even number is an integer that is divisible by two.

Notice that to understand this definition we need to know the definition of the mathematical object 'integer' and the definition of the mathematical property 'to be divisible by'. If we know these two definitions, we can reformulate the previous definition as:
Definition 2: A number z is even if and only if $z/2 \in \mathbb{Z}$.

You see the 'if and only if' in the definition gives a logical equivalence between the expression"z is even" with the precise mathematical formulation "$z/2 \in \mathbb{Z}$".

As you can see from this example, sometimes one has to rework a definition to get to write in mathematical symbols, as we did when we went from definition 1 to definition 2.

Notice further, that one can even rewrite the definition in an equivalent, but slightly different way:
Definition 3: A number z is even if and only if there exists and integer k such that $z = 2k$.

Definitions are your starting point

- If you want to learn to write in a mathematical way, you need to start from the definitions.
- Know definitions with precision. Learn how to express them mathematically (like going from Definition 1 above to Definition 2 or 3).
- When solving exercises, for each mathematical object or property, you can only use their definitions. Anything else is not a mathematical argument.

Exercise. Express symbolically that a number is a multiple of 6.

Exercise. Express symbolically the idea that 'an element belongs simultaneously to two sets'.

Exercise. Let A, B be two sets. An exercise asks as to prove that $A = B$, what do we actually need to show?

Exercise. What is the definition of a *function*?

1.2 Assumptions

- An assumption is an statement p that we take to be true.
- An assumption is not true or false, it is an assumption. However, an assumption can be empty when there is no case under which p is true, e.g. "Suppose x is a real number divisible by 0".
  When this happens, any claim coming from this assumption is useless.

1.3 Claims: lemma, proposition, theorem, corollary

Mathematical properties of mathematical objects are expressed through claims that must be proven using logical arguments. Claims are classified as Lemmas (when they are auxiliary claims used to prove other claims), Propositions and Theorems (which correspond to more important claims, and Corollaries, which are direct consequences of Propositions and Theorems.

1.4 Proofs, logic and mathematical theories

We will see more of this in consequent chapters. What is important here is to remember that new mathematical knowledge is **built upon** previous one. This is why it is essential to have a good mathematical

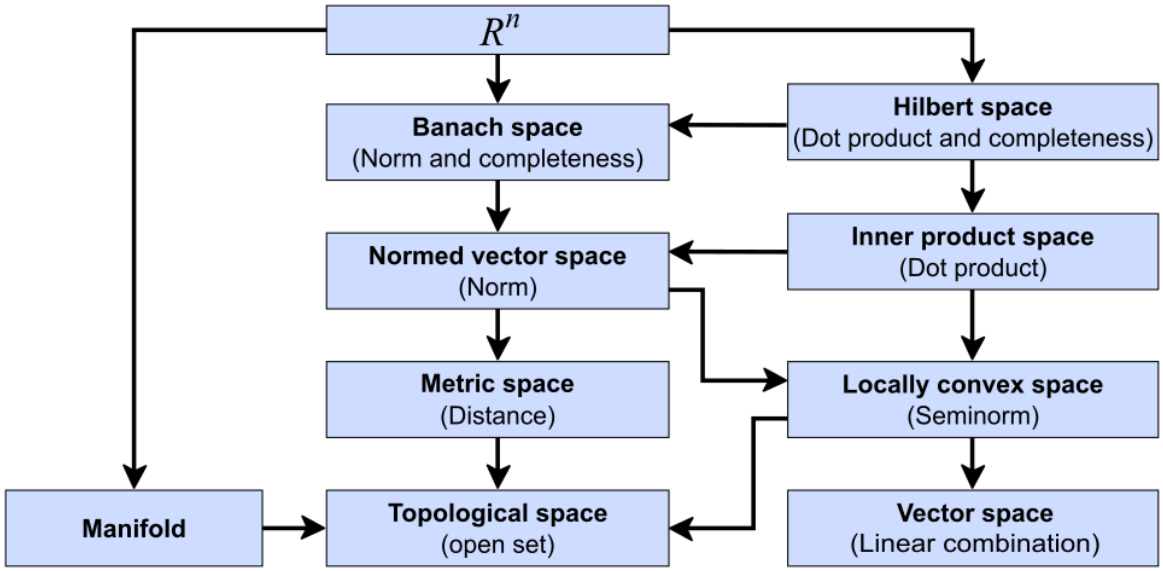


Figure 1.1: An example of mathematical objects built upon other mathematical objects. Source: https://commons.wikimedia.org/wiki/File:Mathematical_implication_diagram-alt-large-print.svg

foundation to be able to pass future courses. So always do your best and learn as much as possible if you do not want to struggle in upcoming semesters. See an example in Fig. 1.1.

Chapter 2

Studying with structure

2.1 Warm up

Take the lecture notes and consider 3-4 pages of what is going to be covered in the next class. Take 15 minutes to prepare the next class. After that, reflect on your method by answering the following questions:

1. Strategy:
    - How did you read? Linearly? Did you give the same attention to everything or did you pay attention to some particular components?
    - Did you take notes or marked down the text? Which type of notes (questions, diagram, highlights, underline) did you take?
    - With which *intention* did you read? Trying to understand each concept? Trying to create a mental map? Trying to understand line by line? Trying to relate to previous material? Trying to memorize?
2. Evaluation:
    - How much do you recall? Try to write it down.
    - How helpful this study has been?
3. Emotions:
    - How did you feel as you were doing this exercise? (bored, excited, anxious, skeptical,...).
    - How focused were you while doing the exercise?

Revise after each class and prepare before the next class. Here you have a typical study plan, which you can modify according to what works best for you.

2.2 After each theory class

Review the content of the class as follows:

1. Read the lecture notes. Identify definitions, examples, comments, properties, and proofs. You can ignore or skim the proofs on the first reading. Prioritise learning definitions before moving to claims.
2. On successive readings, focus carefully on the proofs. Try to predict how the proofs unfold before reading them fully.

3. From these readings:
   - Take notes of anything that is unclear;
   - Create a mind map of key concepts and properties covered - see examples in figures 2.1 and 2.2.
4. Look at the mind map without other notes:
   - Can you write every definition with precision?
   - Can you write the negation of a definition?
   - Can you provide examples of these definitions? (Perhaps from the exercise class?)
   - For every claim, can you write a full statement?
   - Why is the claim important? How does it relate to other concepts? How can it be applied?
   - How does this new theory connect to previous classes and concepts?
   - Do you understand every step of the proof of this claim?
   - Can you sketch the proof and outline the key steps?
   - Is the proof direct, indirect, or by contradiction?
5. Can you see from the mind map how all concepts are related?
6. Review the list of exercises assigned. Identify which theoretical concepts are needed to solve each problem.

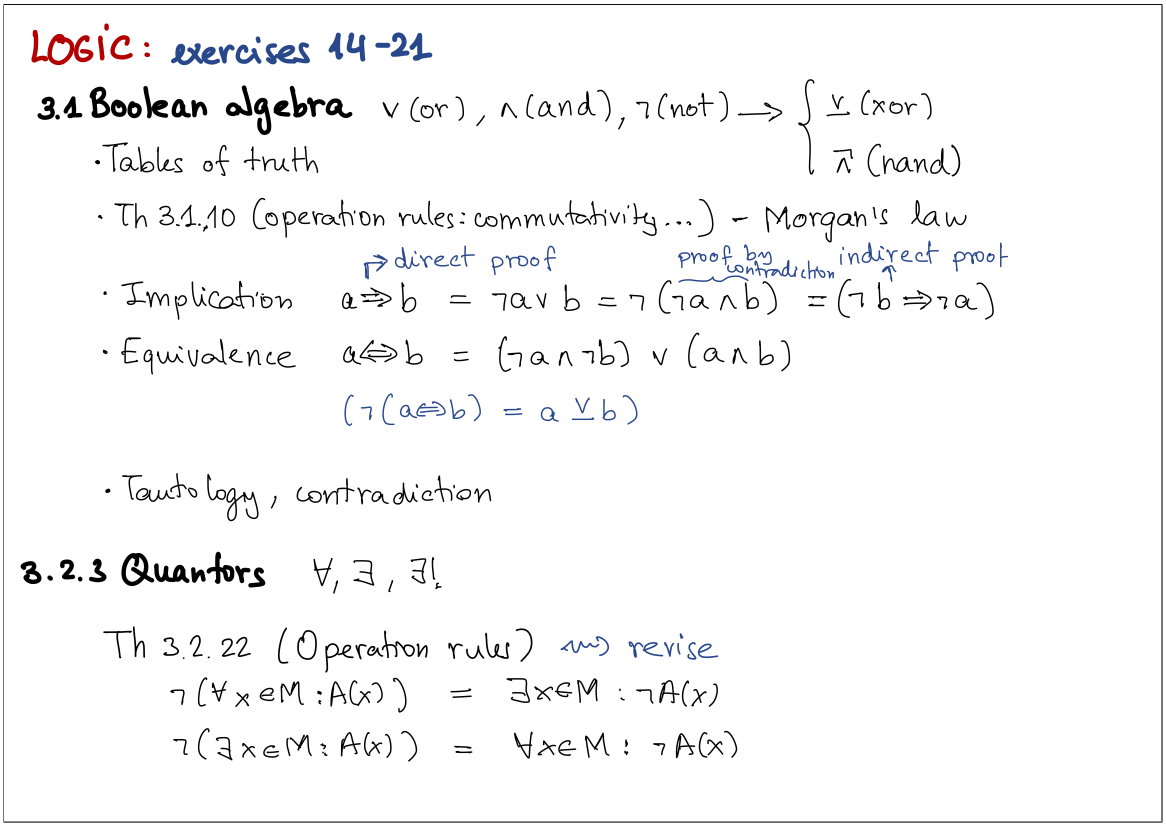


Figure 2.1: Example of a summary from a chapter in Logic. The numbers in '3.1 Boolean algebra' and '3.2.2 Quantors' indicate the sections in the lecture notes where further information can be found. 'Exercises 14-21' at top indicates which exercises cover this topic.

This other examples of a mind map connects different mathematical objects and their properties (from a course on Functional analysis).

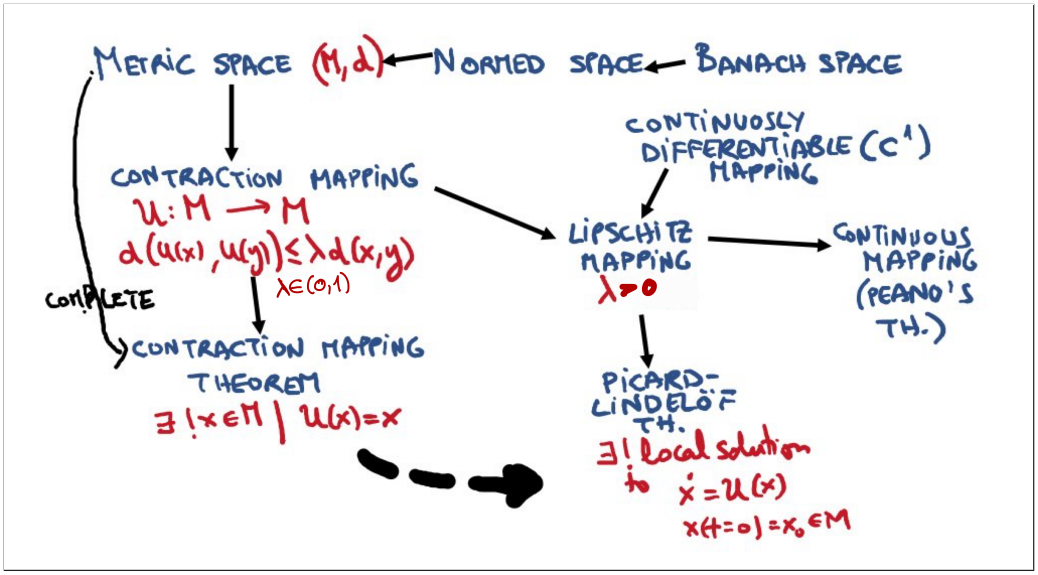


Figure 2.2: An example of a mind-map or diagram as part of a course on Functional Analysis. Here a Banach space is in particular a normed space, which is in particular a metric space. In metric spaces one can define contraction mappings and we have the contraction mapping theorem which is key to proving the Picard-Lindeloef theorem of existence and uniqueness of solutions for Ordinary Differential Equations. At the same time a contraction mapping is a more restrictive form of a Lipschitz mapping, which is the condition needed in the Picard-Lindeloef theorem, etc. **To understand a diagram one must have knowledge of the objects in them and their relations. The diagram gives the big picture of all the elements in a theory and how they are connected.**

2.3 Before each theory class: preparation

- Read the lecture notes in advance. This reading doesn't need to be as detailed as the post-class review.
- Create a preliminary mind map, extracting key points. How do new concepts connect to previously learned material?
- Make your reading active:
  - When encountering a definition, try to think of your own examples before checking the ones given in the notes.
  - If a proof is presented, try predicting how it might work before reading it in full.
  - Read part of the proof and pause—can you anticipate the next step?
- Write down questions. Check during class whether they get answered naturally.

Even if you only devote 15 minutes to reading the lecture notes ahead of the class, it already gives huge benefits. You will understand much more, much faster.

2.4 During theory class

During class, do not copy the blackboard, instead try to understand as much as possible. Copy only when new material, information, or examples are given. Sit there, listen, and try to understand as much as possible. See if your questions are answered during the explanation. Try always to be active and not passive.

Other tips:

- If you start feeling sleepy during the class, remember that you may need to drink water! (and avoid heavy meals before a class!).
- Forget about your phone during the class: do not occupy yourself with anything that is not the class. Train yourself to avoid distractions.

2.5 Preparing for an exercise class

Strategies for problem-solving on the Chapters 4 and 5.

2.6 Revision of an exercise class

- After an exercise class, review the mistakes you made in the exercises. What can you learn from them? Make note of it.
- For exercises where you struggled: reattempt them without looking at the notes. Can you now solve them independently?

2.7 During the exercise class

Pay attention not only to see if your solution is correct, but also use the feedback to learn:

- how to write mathematics;
- how to explain mathematics (see what works and what does not work);
- common mistakes and how are they corrected.

Build your own study material as a collection of solutions that are easy to use in the revision process, in particular:

- Have a clean set of solutions, mark during the class where you made mistakes so that you can revise them easily later.
- Make a note of concepts or properties that you did not understand well and you need to revise. Detect any weak point in your knowledge or logic.

Chapter 3

Problem-solving. Part I: The process

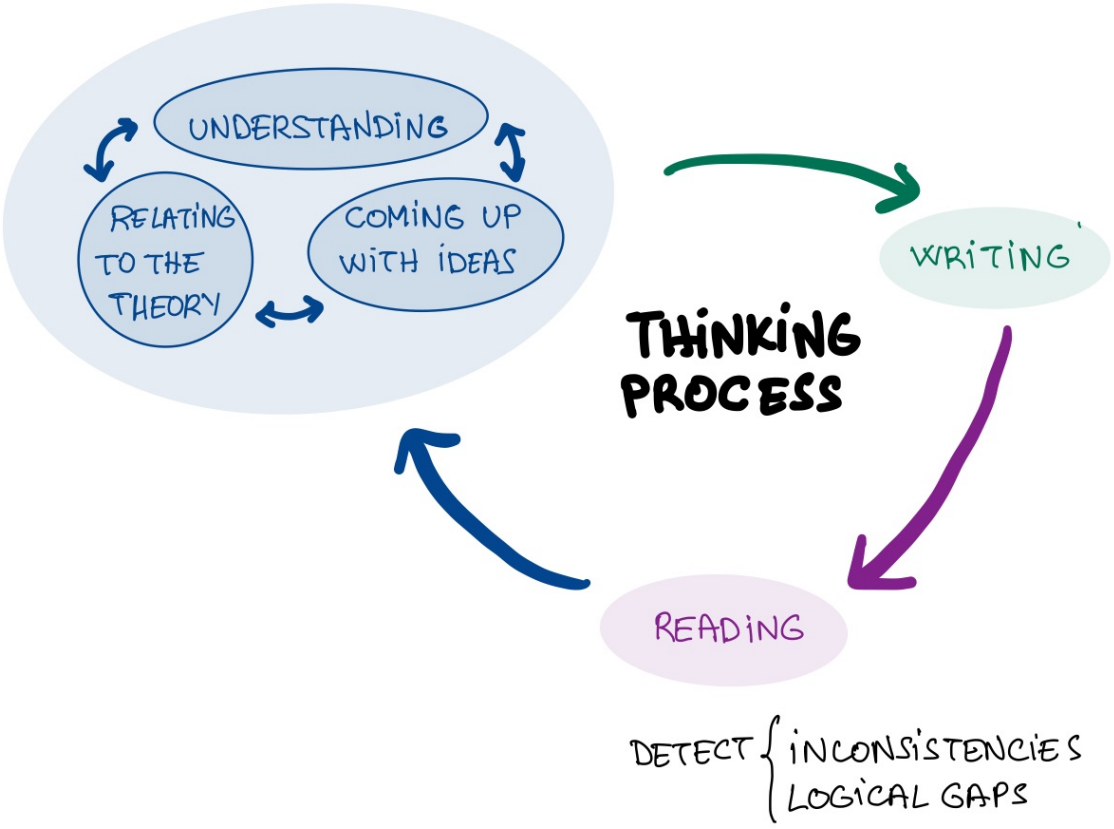


Figure 3.1: Problem solving is an **iterative thinking process** where writing and reading your solutions is an essential part of the process. Writing and thinking are inseparable. That is why need to learn how to write.

Chapter 4

Problem-solving. Part II: 6 steps

When solving problems, one needs a strategy. The following is inspired on Polya's problem-solving strategy from his famous book "How to solve it".

STEP 0. THROW YOUR PEN! AND KEEP SOME DISTANCE

Put down your pen! Resist the urge to start solving right away. Instead, take a moment to pick up your list of exercises and read through them carefully. Don't try to solve them yet—just approach them with curiosity and focus on understanding each one, especially:

1. Understand every word, concept and property mentioned in the problem. Make sure you can write down the **definitions**.
    - While reading the exercise, underline any words you don't know and look up their definitions in your lecture notes. Avoid Googling them or asking ChatGPT—you'll only waste time and risk confusion, as definitions and notation may vary.
2. Identify the theoretical concepts required to solve the exercise. Recognize any gaps in your knowledge and review the relevant theory to fill them before attempting the exercise.

STEP 1. GET FAMILIARISED WITH THE PROBLEM

1. If you can, draw a figure.
2. Introduce suitable notation - especially if there are parts of the problem that are expressed in words rather than with mathematical symbology - use the definition of the objects to help you with this.
3. When dealing with abstract mathematical objects, build concrete examples to familiarize yourself with those objects. Check if what you are trying to prove holds for those particular examples.

STEP 2. DEVISE A PLAN / PLAY WITH IT!

1. Could you restate the problem? Could you restate it still differently? Go back to definitions.
2. Play with it! If you're stuck, take a moment to explore the problem and ask yourself, "What do I wonder?" In other words, what other questions about the objects and properties involved come to mind?

3. If you cannot solve the proposed problem, try to solve first some related problem. Could you imagine a more accessible related problem? A more general problem? A more special problem? An analogous problem? Could you solve a part of the problem? Can you breakdown the problem into easier parts?
4. Which parts of the theory are related to this problem? Are there examples or methods in proofs covered in class that can help solve the problem?
5. Think of similar problems that you already know, could you use them?
6. Did you use all the assumptions? Have you taken into account all essential notions involved in the problem?
7. For exercises requiring you to proof a claim, see Chapter 5 for more details.

STEP 3. WRITE DOWN YOUR SOLUTION

- Write full sentences.
- Write down the logical connection between steps. Indicate where the assumptions are used.
- Write down consecutive versions and for each version aim to become more precise and more complete in your answer.

STEP 4. READ YOUR SOLUTION

- Do a quality control of your solution following Chapter 6 to make sure that you have a good solution.
- Is your written solution understandable to others?
- Can you check the result? Can you check the argument?

STEP 5. ITERATE

If you encountered problems when reading your solution, go back to step 1 and 2 and repeat this process as needed.

STEP 6. THE EXTRA MILE (GETS YOU FURTHER)

1. Reflect on the answer (adapted from Alcock "How to study for a mathematics degree")
    - Why did that procedure work?
    - What could be changed in the question so that it would still work or not work?
    - Can you use the result, the method (or modifications) for some other cases?
    - Can you derive the solution differently? Can you see it at a glance?
2. Stay organised. Ensure your exercises are written clearly and legibly, with each problem statement followed by its solution. Store them in an easily accessible format and organize your work systematically to facilitate the revision process.

Chapter 5

Problem-solving. Part III: Writing proofs, where to start?

Abstract thinking and proofs are why mathematics is so valued. But making proofs is not easy. It takes a lot of time and practice to just learn to read proofs, so it takes even longer to write them. Studying mathematics constantly trains you in reading and writing proofs. So do not try to run before you can walk. Be patient.

- **Read proofs** (learning how to read, precedes learning how to write). Read the lecture notes. Pay special attention to examples or material that relate to the exercise that you are trying to solve.
    1. Can you imitate the approach used in the lecture notes for your exercise?
    2. Can you find tips on how to proceed or formalize parts of the proof you are trying to write?
    3. Put the lecture notes on the side, are you able to reproduce the proof?
- **Start by writing the definitions** of every object and property that appears in the problem. This is the way of introducing mathematical language. Sometimes it is also useful to write the negation of these definitions.
- **Proof types.** Consider different ways of carrying out the proof: direct, indirect, proof by contradiction, by induction, by counterexample,.... Can you write the statement in indirect form? What would be a proof by contradiction?
- **The process.** Writing a proof is a process
    1. Write a first draft.
    2. Read it critically: detect gaps and inconsistencies.
    3. Think about how to fix these.
    4. Revise and repeat until you produce the best version possible.
    5. Sometimes, you may need to rewrite the proof entirely—this is normal!

Emotions. Writing proofs requires full focus. Emotions will generally unfocus you. So if an emotion arises like frustration or excitement, first calm down the emotion.

Exercise on writing proofs[1]

Proof the following theorem:

[1] By Kata Sebök (adapted from Lara Alcock).

Theorem 5.1. *Every prime number (except 2 and 3) is a neighbor of a multiple of 6.*

Instructions:
Follow the same steps as in the instructions for the proof of Theorem 10.5 to arrive at your own proof of the statement. You may need more time here to arrive at a plan for your proof. Don't be afraid to experiment with many concrete examples at the beginning to identify potentially helpful patterns.

Chapter 6

Problem-solving. Part IV: How to know if my solution is correct?

Basic quality control of WRITTEN solutions - a checklist

- Have I applied only definitions of objects and properties, and not vague notions?
- Have you used all the assumptions? Where?
- Have I given a full answer?
- Is it logically consistent? (Everything is well defined; every individual expression is correct; legit application of propositions).
- Is the logical connection between lines clearly stated? (implication, equivalence,...).

6.1 Explained quality control checklist

Before solving an exercise

1. Identify mathematical objects and mathematical properties.
2. Do I know the definition of all the mathematical objects and mathematical properties that appear in the exercise? Can I write them down? Can I give concrete examples of these objects? For example:
   - In exercise 6.1, do I know what 'sets' mean, do I know what 'intersection' $\cap$ means? Do I know what $A \backslash B$ means and also, what is the definition that two sets are equal?
3. Do I know the statement of the propositions and theorems that appear in the exercise? Can I write the statement down in all detail (including the assumptions of those statements)?

Mnemonic rule: **I.D.** your exercise first (IDENTIFY - DEFINE).

After solving an exercise

List of basic standard to be achieved.

1. **LOGICAL LINKS:** Is the logical connection **between lines** clearly stated? (implication, equivalence,...)

2. **DEFINITIONS:** Make sure that you only use definitions of objects and properties to write down your solution.

3. **ASSUMPTIONS:** In my solution, have I used all the assumptions stated in the problem? Have I specified where and how in the argumentation those assumptions are used?

   For example,

   - in exercise 6.2 have I used that p is prime?
   - in the solution of exercise 6.3 have I used that $a \cdot d \neq b \cdot c$?

4. **FULL ANSWER**: Have I given a full answer?

   - Have I considered all the cases in full generality or have I just, unwittingly, proven a concrete case? Or are we forgetting some particular cases?

     - For example, maybe we proved the result only for n being a natural, but n is actually an integer.
     - In exercise 6.1 apart from proving the equality between the sets in equation (6.1), have I also remembered to prove that the sets are pairwise disjoint?
   - Have I really proven what I was asked to prove or something else? Have I proven all that I was asked to prove?
     For example, did we have to proof an equivalence but we only proved one of the two implications?

5. **LOGICAL CONSISTENCY**

   (a) **Everything is well defined.** Does everything that you wrote down makes sense? Or in other words, everything that you wrote down is **well defined**?

   - **Notation:** For example, if A and B are sets, writing $\neg A$, $A \wedge B$, $A - B$ make no sense because $\neg$ and $\wedge$ are symbols used in logic and $-$ is a substraction. When you write those things probably what you mean is A^c (the complementary of A), $A \cap B$, and $A \backslash B$. The first three expressions make no sense, the last three expressions make sense.
   - **Well defined mathematical objects:**
     - The function $[f : [0, 1] \to \mathbb{R}]$ given by
       $$f(x) = 1/x$$
       is not well defined at 0. So 0 has be to excluded from the domain of f.
     - The sum
       $$\sum_{k=1}^{k+n} a_k$$
       is not well defined since we are summing up to $k + n$ but k is the index of summation. The same goes for the integral
       $$\int_0^x f(x)dx,$$
       we cannot have the integration interval $[0, x]$ depend on the integral variable x.
   - **Well defined operations:**
     - For example, if $\lambda \in \mathbb{R}$ and $v, w \in \mathbb{R}^2$, $\lambda + v$ makes no sense because they have different dimensions and vw makes no sense because we cannot multiply two vectors as if they were scalars.

  - if $[f : A \to B]$ and $[g : C \to D]$ are two functions, then does it make sense to consider the composition $f \circ g$ and $g \circ f$? In the first case it is well defined only if $D \subset A$ and the second case, if $B \subset C$.

(b) **Every individual expression is correct.** Is everything that you wrote down correct? (i.e. do equalities, equivalences and implications hold?).

(c) **Legit application of propositions.** If you applied a proposition or a theorem, did you check that the assumptions for that proposition and theorem hold?

The third checklist: the teacher's expectation

When the teacher calls you to the blackboard to present a solution, they expect that you have prepared your written material properly so that you do not have to say sentences like:

- 'I do not understand what I wrote in my notes'.
- 'I do not remember what the exercise was about, do you have the printed exercise sheet?'

Extra note: it gives a very bad impression if you have your solutions only on your cell phone (can you guess why?).

Examples of exercises[1]

Exercise 6.1. *Let A and B be sets. Show: The sets $A\backslash B$, $A \cap B$, and $B\backslash A$ are pairwise disjoint, and it holds that*

$$A \cup B = (A\backslash B) \cup (A \cap B) \cup (B\backslash A). \tag{6.1}$$

Exercise 6.2. *Show: If p is a prime number, then for all $a, b \in \mathbb{Z}_p$ it holds:*

$$(a + b)^p = a^p + b^p.$$

Exercise 6.3. *Let $a, b, c, d \in \mathbb{R}$ be four real numbers with $c \neq 0$ and $a \cdot d \neq b \cdot c$. Consider the (real-valued) function defined as*

$$f(x) = \frac{a \cdot x + b}{c \cdot x + d}.$$

Show:

(1) The domain of f can be chosen as $\mathbb{R}\backslash\left\{-\frac{d}{c}\right\}$,

(2) For this domain, $img(f) = \mathbb{R}\backslash\left\{\frac{a}{c}\right\}$.

Show furthermore: The function

$$[f : \left(\mathbb{R}\backslash\left\{-\frac{d}{c}\right\}\right) \to \left(\mathbb{R}\backslash\left\{\frac{a}{c}\right\}\right)]$$

has an inverse function. Compute the functional expression of this inverse function.

Exercise 6.4. *Let $(r_n)_{n\in\mathbb{N}}$ be a monotone decreasing sequence of real numbers, which converges to limit ρ, i.e.*

$$\lim_{n\to\infty} r_n = \rho.$$

Show: For all $n \in \mathbb{N}$ there holds $r_n \geq \rho$.

[1] *Note: this list of exercises are taking from the STEOP course that took place in Winter 2024, by Markus Fulmek.*

6.2 Types of mathematical errors

TYPE OF MISTAKE	EXAMPLES	POTENTIAL SOURCES
1. False statements	- A definition is wrongly stated (something is missing or is incorrect). - Computational errors. - Theorem wrongly stated.	- Laziness (e.g. not writing a new line and keeping computations on the same line). - Not knowing how to write maths. - Not checking. - Not knowing how to read maths. - Not knowing the theory well.
2. Absolute non-sense: Invalid operations, non-sensical statements, empty or undefined statements	- Invalid operation (e.g. multiply two vectors, dividing by zero,...) - Empty definitions, e.g. 'Let $k > 0$ be such that $-3k > 1$.' - Invalid properties, e.g 'Let $v \in \mathbb{R}^2$ be such that $v > 0$.' - Non-sensical statements, e.g. $\sum_{k=1}^{2k+1} a_k$	- Forgetting (or not knowing) what each object/operation is (and its properties). - Not checking the validity of definitions/operations.
3. Gaps in an argument	- A step in a proof is not justified - Assumptions of a theorem are not checked. - The conditions of a definition are not checked. - An object is used but not defined.	- Lack of revision - Not knowing how to justify the gap. - Not knowing how to apply a theorem. - Not knowing a definition.
4. Partial answer or proof (a case is omitted)	- Instead of proving an equivalence, only one of the implications is proven. - Particular cases are not checked.	- Lack of revision. - Not checking for special cases.
5. Logical errors (Part I)	- Not checking the base case in an induction argument. - Assuming what you want to prove to prove it (circular argument). - Using an implication as if it was an equivalence. - Wrong negation of a statement.	- Being weak in logic. - Poor writing. - Poor exercise preparation. - Poor revision.
6. Logical errors (Part II) (getting implications wrong)	- Confusing between 'necessary' conditions and 'sufficient' conditions. - Trying to proof that a claim is false instead of looking for a counterexample. - Just give a particular example for a statement and consider it a proof. - To think that if you remove an assumption from a theorem, then the conclusion of the theorem cannot be any more true.	- Being weak in logic. - Biases
7. Typos	- Change of notation along an answer. - Copying wrongly the exercise.	- Lack of revision

Some errors are fixable, some are not.

Chapter 7

Logic Part I. Implications

Note: The chapters in logic require basic knowledge of tables of truth, the logical operators 'and' ($\wedge$), 'or' ($\vee$, and 'not' ($\neg$), as well as quantifiers (see summary in Fig. 2.1).

The following rules applies to these cards:

If a card has a vowel on one side, then it has an even number on the other side.

Answer as quickly as possible:

Which 2 cards do you need to turn over to check the rule for correctness?

We have many cognitive biases, which include biases on how we think about things. For example, there is a well-known bias named "confirmation bias" (Nickerson, 1998). In this bias, we tend to look for confirmation or further support of some prior knowledge or experience, but we do not tend to look for counter-examples[1].

[1] Visit the full course at https://www.uni-konstanz.de/unconscious-bias-tool/

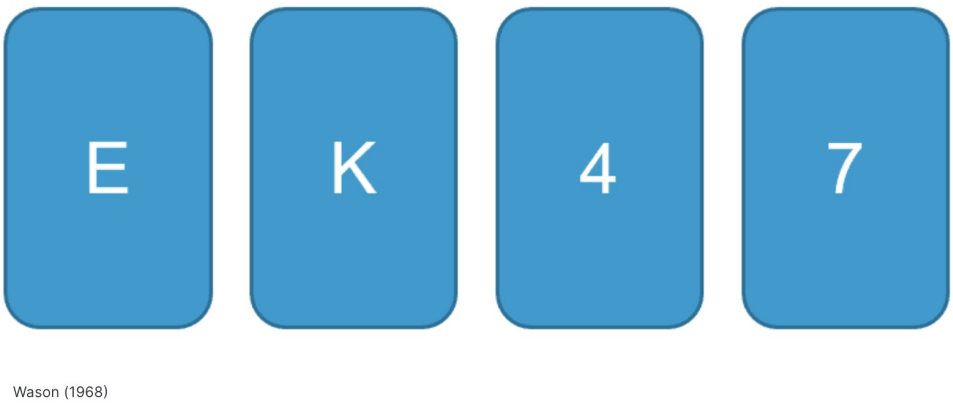


Figure 7.1: From the online course on cognitive bias by the University of Konstanz, which is taken from a study by Wason (1968).

Implications

Given two statements p and q, we say "p implies q" (in logical notation $p \Rightarrow q$) when we have the following table of truth:

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Notice that the implication $p \Rightarrow q$ is false only when q is true and p is false.

Equivalent ways of writing the implication $p \Rightarrow q$

$\neg(p \wedge \neg q)$: used in proofs by contradiction
$\neg p \vee q$
$\neg q \Rightarrow \neg p$: used in proofs by contrapositive

Exercise: show that $p \Rightarrow q, \neg(p \wedge \neg q), \neg p \vee q, \neg q \Rightarrow \neg p$ are all equivalent

Proving the implication $p \Rightarrow q$: types of proof

Type of proof	Form of the implication	Assumptions to be made	What needs to be proven
Direct proof	$p \Rightarrow q$	p	q holds
Proof by contrapositive	$\neg q \Rightarrow \neg p$	$\neg q$	$\neg p$ holds
Proof by contradiction	$\neg(p \wedge \neg q)$	$p \wedge \neg q$	we reach a contradiction

Exercise:

Exercise 7.1. *the implication in the form $\neg p \vee q$ is less used in proofs, in this case, what would be the assumptions and what would one need to prove? Hint; when is 'or' ($\vee$) true?*

Common mistake: Instead of $p \Rightarrow q$, take as true $q \Rightarrow p$ or $\neg p \Rightarrow \neg q$
Exercise: For each one of the next two statements prove them if true, or find a counterexample if wrong.

- If $p \Rightarrow q$, then $q \Rightarrow p$.
- If $p \Rightarrow q$, then $\neg p \Rightarrow \neg q$

Necessary and sufficient conditions Suppose that $p \Rightarrow q$. We say that

- p is a **sufficient condition** for q;
- q is a **necessary condition** for p.

Example

- Let $p =$ "$n \in \mathbb{N}$ is divisible by 4"; $q =$ "$n \in \mathbb{N}$ is divisible by 2", then $p \Rightarrow q$. So a number being divisible by 4 is sufficient to conclude that it is also divisible by 2. And, a number being divisible by 2 is necessary for it being divisible by 4, in the sense $\neg q \Rightarrow \neg p$ (but it is not sufficient since $q \not\Rightarrow p$).
- Some conditions are not necessary nor sufficient e.g. if $p =$ " $n \in \mathbb{N}$ is even"; $q =$ " $n \in \mathbb{N}$ is prime", then p is not necessary nor sufficient for q.

Exercise: If p is necessary and sufficient for q, what does this mean?

Exercise. A fellow student gives you this solution. Analyse it.
Task: Find the solution (or solutions) $(x, y) \in \mathbb{R}^2$ to

$$axy = c \tag{7.1}$$
$$bxy^2 = d \tag{7.2}$$
$$x + cy = 0 \tag{7.3}$$

for $a, b, c, d \in \mathbb{R}$.
Given solution:
From (7.3) we have that $x = -cy$.
Substituting in (7.1) we get that

$$a(-cy)y = c \Leftrightarrow -acy^2 = c \quad \Leftrightarrow \quad y^2 = a \quad \Leftrightarrow \quad y = \pm\sqrt{a}$$

So $(x = -cy, y = \pm\sqrt{a})$ are the solutions.

Chapter 8

Logic Part II. Negation

Note: some of the exercises are from the exercise class done by Andreas Cap and Hermann Schichl during the STEOP course, summer 2025.

8.1 Negating definitions

Consider the definitions below and negate them.

Definition 8.1 (Subsets)**.** *Let A, B be sets. We say that A is a subset of B, denoted $A \subset B$, if and only if,*

$$\forall x,\ x \in A \Rightarrow x \in B.$$

Definition 8.2 (Intersection of sets)**.** *Let A and B be sets. The **intersection** of A and B, denoted by $A \cap B$, is the set of all elements that belong to both A and B. Formally:*

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\} \tag{8.1}$$

Definition 8.3 (Union of sets)**.** *Let A and B be sets. The **union** of A and B, denoted by $A \cup B$, is the set of all elements that belong to either A or B (or both). Formally:*

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\} \tag{8.2}$$

Definition 8.4 (Convergent Sequence)**.** *A sequence of real numbers $(a_n)_{n=1}^{\infty}$ is said to **converge** to a real number L if for every $\varepsilon > 0$, there exists a positive integer N such that*

$$|a_n - L| < \varepsilon \quad \text{for all} \quad n \geq N. \tag{8.3}$$

8.2 Using the exercises to practice logic

THE CREAM OF IT ALL! Make sure that you are fluent with concepts, properties and **logic** - **using indirect thinking and negations**.

1. If the exercise is about proving an implication, do I know how to write the indirect implication?
    - Exercise 8.5 is tricky in this sense. Notice that there are three statements relating to the sequence of real numbers $(r_n)_{n\in\mathbb{N}}$, let's call these statements p, q and r, which are:

      p = " $(r_n)_{n\in\mathbb{N}}$ is a monotone decreasing";
      q = "it holds that $\lim_{n\to\infty} r_n = \rho$";

r = "for all $n \in \mathbb{N}$ there holds $r_n \geq \rho$".

We are asked to prove

$$\text{if } p \text{ and } q, \text{ then } r.$$

(or in notation form logic $p \wedge q \Rightarrow r$). The indirect statement corresponds to

$$\text{if } r \text{ is not true, then it is not true that } p \text{ and } q$$

(or in logic notation $\neg r \Rightarrow \neg(p \wedge q)$). And now we need to negate all this statements:

$\neg r$ = **there exists** $n_0 \in \mathbb{N}$ such that $r_n < \rho$.

$\neg(p \wedge q)$ = it cannot be that simultaneously p is true and q is true. In other words, either one of them is false, or both are false (remember that

$$\neg(p \wedge q) \text{ is logically equivalent to } \neg p \vee \neg q,$$

which means that one of the two statements, p or q must be false. Now we negate the statements p and q separately. To negate them, first we need to write precisely the definitions of the properties that appear in these statements, i.e., we need to go to the definitions of being a 'monotone decreasing sequence' and 'convergence of a sequence'

p = for all $n \in \mathbb{N}$, it holds $r_n \geq r_{n+1}$

$\neg p$ = there exists $n_0 \in \mathbb{N}$ such that $r_{n_0} < r_{n_0} + 1$;

q = $\forall \varepsilon$, $\exists N_\varepsilon \in \mathbb{N}$ such that $\forall n \geq N_\varepsilon$ it holds $|r_n - \rho| < \varepsilon$.

$\neg q$ = there exists ε_0 such that for all $N \in \mathbb{N}$, there exists $n_0 > N$ such that $|r_{n_0} - \rho| > \varepsilon_0$

2. If the exercise is based on or uses a property, do I know how to write down that that property does not hold? [Example, negation that a function is injective; negation that a sequence converges; negation that a function is not linear; negate that two sets are equal]

Let's practice now

Consider the list of exercises below

1. What is the negation of the statement in exercise 8.1.
2. What is would be the statement of exercise 8.2 in indirect from. What would be the statement if we wanted to do a proof by contradiction? Do the same for 8.4.
3. Regarding exercise 8.3, what is the indirect proof regarding the part f is injective. How would the statement be written if it was a proof by contradiction?
4. Write exercise 8.5 in indirect form.

Exercise 8.1. *Prove that for every natural number $n \geq 2$, the number $n^3 - n$ is divisible by 6.*

Exercise 8.2. *Let A, B, and C be arbitrary sets. Prove: If $A \subseteq B$, then $A \cap C \subseteq B \cap C$ and $C \setminus A \supseteq C \setminus B$.*

Exercise 8.3. *Let $f : X \to Y$ be a function and let $A_1, A_2 \subseteq X$ be subsets. Prove that $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$ and that equality holds if f is injective, but not in general.*

Exercise 8.4. *Let $f : X \to Y$ and $g : Y \to Z$ be functions and $g \circ f : X \to Z$ their composition. Prove:*

(a) If f and g are surjective, then $g \circ f$ is surjective.

(b) If $g \circ f$ is surjective, then g is surjective.

Find an example where in situation (b), f is not surjective.

Exercise 8.5. *Let $(r_n)_{n\in\mathbb{N}}$ be a monotone decreasing sequence of real numbers, which converges to limit ρ, i.e.*

$$\lim_{n\to\infty} r_n = \rho.$$

Show: For all $n \in \mathbb{N}$ there holds $r_n \geq \rho$.

Chapter 9

Logic Part III. Types of proof

The most common types of proof are:

- Direct proof.
- Proof by contrapositive.
- Proof by contradiction.
- Induction proof.
- Proof by counterexample.
  - To show that a claim is true, we need a proof. To show that a claim is false, we need a counterexample.

Exercise. True or false?
For each claim below, if it is true, prove it, if false, give a counter-example.
Let $a \in \mathbb{R}$ be such that

$$a < \sqrt{b}$$

for $b > 0$, then it holds that
i) $a^2 < b$,
ii) $a^2 < a\sqrt{b}$,
iii) $a^2\sqrt{b} < b$.

Exercise Look at the following statements. For each, if true, prove it by induction, if false, give a counter-example.

i) For all $n \in \mathbb{N}, n \geqslant 1, a, b > 0$, is it true that

$$(a+b)^n = a^n + b^n + (n-1)\left(\frac{a^{n-1}+b^{n-1}}{2}\right) ?$$

ii) for $n \geqslant 1$, is it true that

$$\sum_{k=1}^{n+1}(2k-1) = n^2 + 3 ?$$

Chapter 10

Reading maths

10.1 Reading computations

Note: This exercise is translated from the book 'Einfuehrung in das mathematische Arbeiten' by Schichl and Steinbauer.

Theorem 10.1 ((Calculation Rules for Operators of Boolean Algebra)). *For the operations $\wedge$, $\vee$, and $\neg$, the following calculation rules apply:*

1. ***Commutative Laws:*** $a \vee b = b \vee a, \quad a \wedge b = b \wedge a$
2. ***Associative Laws:*** $a \vee (b \vee c) = (a \vee b) \vee c, \quad a \wedge (b \wedge c) = (a \wedge b) \wedge c$
3. ***Distributive Laws:*** $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c), \quad a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
4. ***Absorption Laws:*** $a \vee (b \wedge a) = a, \quad a \wedge (b \vee a) = a$
5. ***Idempotent Laws:*** $a \vee a = a, \quad a \wedge a = a$
6. ***Identity Laws:*** $a \vee 0 = a, \quad a \wedge 1 = a$
7. ***Domination Laws:*** $a \vee 1 = 1, \quad a \wedge 0 = 0$
8. ***Complement Laws:*** $a \vee \neg a = 1, \quad a \wedge \neg a = 0$
9. ***Duality Laws:*** $\neg 0 = 1, \quad \neg 1 = 0$
10. ***Double Negation Law:*** $\neg(\neg a) = a$

11. ***De Morgan's Laws:*** $\neg(a \vee b) = \neg a \wedge \neg b, \quad \neg(a \wedge b) = \neg a \vee \neg b$

Exercise. Justify every equality by giving the number (or numbers) of the 'Computation rules' from Th. 10.1 that are used.

$$\begin{aligned} f(a,b,c) &= (\neg a \wedge \neg b \wedge \neg c) \vee (\neg a \wedge b \wedge \neg c) \vee (\neg a \wedge b \wedge c) \\ &\qquad \vee (a \wedge \neg b \wedge \neg c) \vee (a \wedge b \wedge c) \\ &= (\neg a \wedge ((\neg b \wedge \neg c) \vee (b \wedge \neg c) \vee (b \wedge c))) \\ &\qquad \vee (a \wedge \neg b \wedge \neg c) \vee (a \wedge b \wedge c) \\ &= (\neg a \wedge (((\neg b \vee b) \wedge \neg c) \vee (b \wedge c))) \vee (a \wedge \neg b \wedge \neg c) \vee (a \wedge b \wedge c) \\ &= (\neg a \wedge ((1 \wedge \neg c) \vee (b \wedge c))) \vee (a \wedge \neg b \wedge \neg c) \vee (a \wedge b \wedge c) \\ &= (\neg a \wedge (\neg c \vee (b \wedge c))) \vee (a \wedge \neg b \wedge \neg c) \vee (a \wedge b \wedge c) \\ &= (\neg a \wedge \neg c) \vee (\neg a \wedge b \wedge c) \vee (a \wedge \neg b \wedge \neg c) \vee (a \wedge b \wedge c) \\ &= (\neg a \wedge \neg c) \vee (a \wedge \neg b \wedge \neg c) \vee (\neg a \wedge b \wedge c) \vee (a \wedge b \wedge c) \\ &= (\neg a \wedge \neg c) \vee (a \wedge \neg b \wedge \neg c) \vee ((\neg a \vee a) \wedge (b \wedge c)) \\ &= (\neg a \wedge \neg c) \vee (a \wedge \neg b \wedge \neg c) \vee (1 \wedge (b \wedge c)) \\ &= ((\neg a \vee (a \wedge \neg b)) \wedge \neg c) \vee (b \wedge c) \\ &= ((\neg a \vee a) \wedge (\neg a \vee \neg b) \wedge \neg c) \vee (b \wedge c) \\ &= (1 \wedge (\neg a \vee \neg b) \wedge \neg c) \vee (b \wedge c) \\ &= ((\neg a \vee \neg b) \wedge \neg c) \vee (b \wedge c) \\ &= (\neg(a \wedge b) \wedge \neg c) \vee (b \wedge c) \\ &= \neg((a \wedge b) \vee c) \vee (b \wedge c). \end{aligned}$$

What do we learn from doing this exercise?

- Reading takes time (a long time).
- Every step in a proof must be justified.
- Reading maths is part of the process of learning maths.
  - For example, from reading this proof one understands better the distributivity rule in logic and becomes more fluent in applying it.
- **The goal is not to memorise the proof, but to understand how it works.**

In conclusion, reading mathematics is already a mathematical exercise. It is like solving problems.

What do we NOT cover in this exercise?

The proof considered is simpler than other proofs, particularly, for most of the proofs:

- There are some steps that are skipped (compression) and it is left to the reader to elaborate on these missing steps.
- Some steps are carried out or objects are defined and the purpose that they serve in the proof is not evident (i.e., why are they there?).
- There are no logical steps (the proof consists of just applying rules).

In this exercise, we do not see these two previous aspects. We will explore this in the next section.

10.2 Reading theorems

These are the types of questions that you need to ask yourself when reading proofs:

For each step of the proof, specify:

- Why is the step true?
- Why is this step in the proof? (i.e. what is its purpose?)
- Distinguish between definitions, claims and assumptions,
- For each definition, (i) evaluate if the definition is valid, and (ii) what is its purpose? (why is that particular object defined?)

Reading exercise 1

Theorem 10.2. $\sqrt{2} \notin \mathbb{Q}$.

Proof. Assume that $\sqrt{2}$ is rational.

Then there exist coprime integers $m, n \in \mathbb{N}$ such that

$$\sqrt{2} = \frac{m}{n}.$$

Squaring both sides gives

$$2n^2 = m^2.$$

Since m^2 is even,

it follows that m is also even,

so we can write $m = 2k$ for some $k \in \mathbb{N}$.

Substituting this into the equation, we get

$$2n^2 = 4k^2.$$

Dividing by 2, we obtain

$$n^2 = 2k^2.$$

This shows that n^2 is also even, which implies that n is even.

However, this contradicts the assumption that m and n are coprime.

□

What do we learn from doing this exercise?

- One needs to distinguish between definitions, claims and assumptions.
- Many claims follow:
  - directly from definitions;
  - from direct computations;
  - from logical argumentation (logical contradiction, in this case).
- Some claims may not be so obvious and one may need to check them.
- Every object defined has a purpose.
- Every step of the proof has a purpose.
- Every claim of the proof is justified, i.e., every claim is true and it is shown why it is true.
- The proof is concise.

- When reading the proof, it is not obvious the purpose of some steps and definitions.

What do we NOT cover in this exercise?

- The application of theorems and propositions in a proof.

Reading exercise 2[1]

Theorem 10.3. *There are infinitely many prime numbers.*

Proof. For an arbitrary finite set $\{p_1, \ldots, p_r\}$ of prime numbers, let $n := p_1 p_2 \cdots p_r + 1$.

There is at least one prime divisor of n, let us call it p.

We see that p is different from all p_i, because otherwise p would divide both the number n and the product $p_1 p_2 \cdots p_r$.

Thus, p would also divide 1, which is not possible.

A finite set $\{p_1, \ldots, p_r\}$ can therefore never be the set of all prime numbers.

Hence, there are infinitely many prime numbers.

□

Definition 10.1 (Definition of prime divisor)**.** *A number $p \in \mathbb{N}$ is called a prime divisor of the number $q \in \mathbb{Z}$ if: $p \mid q$ and p is prime.*

[1] Exercise by Kata Sebök (adapted from Lara Alcock).

Reading exercise 3[2]

Theorem 10.4. *There is no smallest positive real number.*

Proof. Assume there exists a smallest positive number $r \in \mathbb{R}$.

For every positive number $s \in \mathbb{R}, s \neq r$, it holds that $0 < r < s$.

Consider $m = \frac{r}{2}$.

It holds that $m \in \mathbb{R}$ and $0 < m < r$.

This leads to a contradiction.

Therefore, there is no smallest positive real number.

□

Reading exercise 4[3]

Definition 10.2 (Abundant number)**.** *An abundant number is a positive integer n whose divisors sum to more than $2n$. For example, 12 is abundant because $1 + 2 + 3 + 4 + 6 + 12 > 24$.*

Theorem 10.5. *The product of two distinct prime numbers is not abundant.*

Proof. Let $n = p_1 p_2$ (where p_1, p_2 are distinct prime numbers).

[2] Exercise by Kata Sebök (adapted from Lara Alcock).
[3] Exercise by Kata Sebök (adapted from Lara Alcock)

Assume that $2 \leq p_1$ and $3 \leq p_2$.

The divisors of n are $1, p_1, p_2$, and $p_1 p_2$.

Note that $\frac{p_1+1}{p_1-1}$ is a monotonically decreasing function in p_1.

Thus, $\max\left\{\frac{p_1+1}{p_1-1}\right\} = \frac{2+1}{2-1} = 3$.

Therefore, $\frac{p_1+1}{p_1-1} \leq p_2$.

Hence, $p_1 + 1 \leq p_1 p_2 - p_2$.

Thus, $p_1 + 1 + p_2 \leq p_1 p_2$.

Finally, $1 + p_1 + p_2 + p_1 p_2 \leq 2 p_1 p_2$.

□

Chapter 11

Things that you may (NOT) take for granted

You may assume that:

- If an exercise asks us to prove a claim, then the claim is true (but the teacher could have made a mistake).
- If an exercise asks us to prove a claim and it makes an assumption, then the assumption is needed to prove the claim (but the teacher could have made a mistake).
- An exercise can be solved only with the material provided during the course or previous courses (but the teacher could have made a mistake).

Notice that the assumptions above are true only in a class context, when doing mathematics in general (for example, when you ask your own questions or when doing research), none of the above apply.

You may NOT assume that:

- All the assumptions in a theorem are necessary, in the sense that if I remove an assumption, the theorem is no longer true.
  - This is not necessarily true. Sometimes we may add an assumption that could be removed just because the proof is simpler.
- Every given definition is non-empty. For example, when we write "Let $n \in \mathbb{N}$ such that $2n > 3$", we are making a claim that there exists such an n.
  - One should check that a definition is well-defined.
- If a teacher gives me a pass in an exercise, it means that I did the exercise well.
  - This is wrong! Humans negotiate, mathematics do not. A mathematical statement or the solution of a problem is correct or incorrect. There are no in-between. However, the teacher may still pass you based on other than your answer being correct.
  - Consider that math is tough and many teachers will try to encourage you. You should not take this as 'I am doing fine', but as 'I am on the right path and I am allowed to make mistakes'.
- It the result is true, even if I did not justify it, it is still ok.
  - For example, to prove a claim I use the mean-value theorem, but the mean-value theorem only applies to continuous functions. You did not check that your function is continuous, but used the theorem anyway. Actually, it turns out that the function was continuous, so the outcome is true.

You may feel satisfied with it, but the argument is completely wrong. You just got lucky. Because your exercises are well prepared, you will get artificially lucky many times (of course, if in class you are giving continuous functions, all the examples then happen to be continuous functions), but this is the reason why the teacher will fail you.

Chapter 12

Writing math

Remember to write full sentences.

Here are some typical structures for sentences commonly used:

1. Definitions
   - **Let** x be a real number **such that**...
   - **Consider** x to be a real number **such that**...
   - **Define** f a function **with the following properties**

   The expressions 'such that' and 'with the following properties' are equivalent.
2. Assumptions
   - Let's assume that... / Assume that...
   - Suppose...
3. Logical sequences
   - You have proven the implication $p \Rightarrow q$ is *true*, you can write:
     - Therefore, from p **follows** q
     - Therefore, q **follows** from p
     - Therefore, p **implies** q.
     - We have **shown** that if p **holds**, then q.
     - If p, then q.
   - To indicate that we are *applying* the implication $p \Rightarrow q$
     - **Since** p, then q.
   - To express the equivalence $p \Leftrightarrow q$
     - Both expressions are equivalent.
   - Why something is true? Since, thanks, by,...
     - q is true, **thanks to** p
     - **By** theorem X, we have that...
     - **By** the previous equation, it holds that/ we conclude that...
     - **By** assumption
     - **Since** p holds, we have that...

  - ... where **we applied** proposition Y...
  - p is true. Therefore, we conclude...
  - p is true. Then, it follows that..
- What do we conclude? Therefore, then, we conclude, it follows, this shows, thus, hence, finally, we obtain.
- Starting sentences: We notice that, we compute,...
- To indicate that we start a new sentence with no logical connections with the previous sentence (probably because we need to do a side computation or a side argument).
  - **Now**, we also have that...
  -

Advise: to start with, just write all the time 'therefore', 'we conclude'.

Exercise. Look back at the first exercises on reading from Chapter 10. Notice that the text is written with full sentences. Notice how the logical connection between sentences is made explicit.

Part II

The mind
(Self-management)

Chapter 13

Reactivity vs Proactivity

People are disturbed not by things, but by the views which they take of things. - Epictetus (a stoic philosopher)

There is nothing either good or bad, but thinking makes it so - Shakespeare in Hamlet.

If you are pained by any external thing, it is not this thing that disturbs you, but your judgment about it. And it is in your power to wipe out this judgment now. - Marcus Aurelius, Meditations.

A man is not hurt so much by what happens, as by his opinion of what happens. - Montaigne

We are more often frightened than hurt; and we suffer more from imagination than from reality. - Seneca

It all depends on how we look at things, and not on how they are in themselves. - Carl Jung

It's not what happens to you, but how you react to it that matters. - Epictetus

Everything can be taken from a man but one thing: the last of the human freedoms — to choose one's attitude in any given set of circumstances, to choose one's own way. - Viktor Frankl

13.1 Under control

Some things are in our control and others not. Things in our control are opinion, pursuit, desire, aversion, and, in a word, whatever are our own actions. Things not in our control are body, property, reputation, command, and, in one word, whatever are not our actions. The things in our control are by nature free, unrestrained, unhindered; but those not in our control are weak, slavish, restrained, belonging to others. Remember, then, that if you suppose that things which are slavish by nature are also free, and that what belongs to others is your own, then you will be hindered. You will lament, you will be disturbed, and you will find fault both with gods and men. But if you suppose that only to be your own which is your own, and what belongs to others such as it really is, then no one will ever compel you or restrain you. Further, you will find fault with no one or accuse no one. You will do nothing against your will. No one will hurt you, you will have no enemies, and you not be harmed. — Epictetus, Enchiridion and Selections from the Discourses

Epictetus is a philosopher that belongs to the group knows as 'stoics'.

Exercise. Read the first case of Katie Byron's *'Loving what is'*. - Introduction to the 4 questions method.

13.2 Proactivity: exercises

Exercise: Read Chapter on Habit 1: Proactivity, of the book 'The 7 habits of highly effective people' by Stephen Covey.

Exercise: From reading the chapter, interpret the diagram in Fig. 13.1:

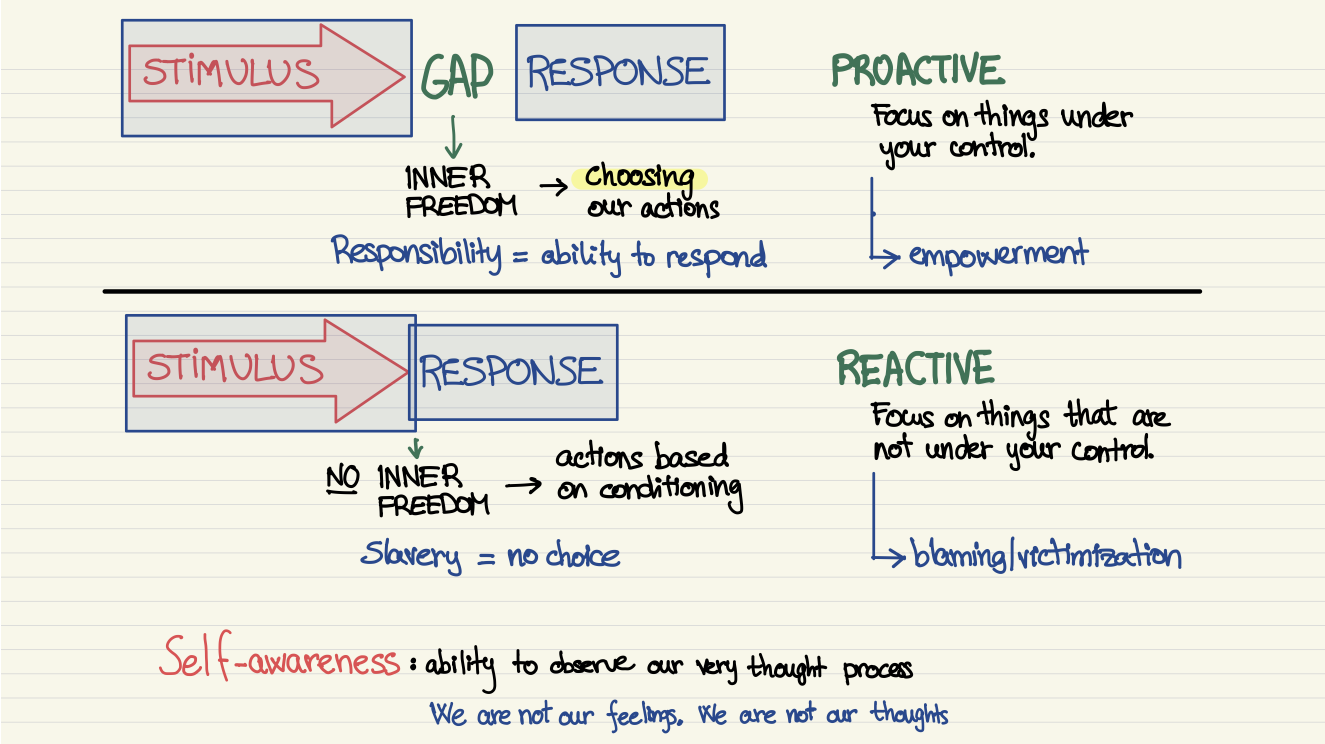


Figure 13.1: Diagram on Proactivity.

Exercise.

- Who is responsible for your posture?
- Who is responsible for the way you eat?
- Who is responsible for your emotions?
- Who is responsible for your likes and dislikes?
- Who is responsible for your manners?

Concrete examples

- You start to feel frustrated while attempting exercises.
  - Who is responsible for your frustration? The teacher? The exercises?

  - Does frustration benefit you or hinder you?

- You ask a colleague for help because you do not know something. Your colleague behaves very arrogantly because it is 'very easy'.
  - This makes you feel stupid and you start to feel that you do not belong here. Who is responsible for these feelings? Your colleague? The teacher that did not encourage you enough?
  - You get angry at your colleague and think that he/she is an asshole. Who is responsible for your anger? Does it benefit you?
- You present an exercise and your solution turns out to be wrong. How do you react and how do you respond?
- You present an exercise and your solution is right. The teacher praises your performance.
  - You start to think that you are better than your colleagues.
  - Who is responsible for this interpretation?
  - Does it benefit you?
- You are not able to stick to your study plan.
  - You start to feel bad about yourself, you blame yourself, you belittle yourself.
  - Who is responsible? The high-school system for not teaching you self-discipline? The teacher from 'Mastering Math and Mind' who was supposed to teach you self-management?
  - Does it help you?
  - **You start to feel bad about feeling bad, because this is only making things worse.**
    * Who is responsible for this?
    * Does it benefit you?
- Someone insults you, you insult back, you shout.
  - Who is responsible for your insults and your shouting?

Exercise. This will be done in class. Listen to a recording by Katie Byron. It will provide a good example in distinguishing between things that are under control (things that our 'our business') and things that are not under our control (things that are 'someone else's business').

13.3 On conditioning

This article provides a good example of how we are conditioned to think, feel and act in a given way:

"Picture yourself as a stereotypical male" by Michelle G. at
https://mitadmissions.org/blogs/entry/picture-yourself-as-a-stereotypical-male/

13.4 Nobody makes it alone

We need to build a support system for ourselves. See Fig. 13.2 to see an example of how people's background affect their paths in higher education.

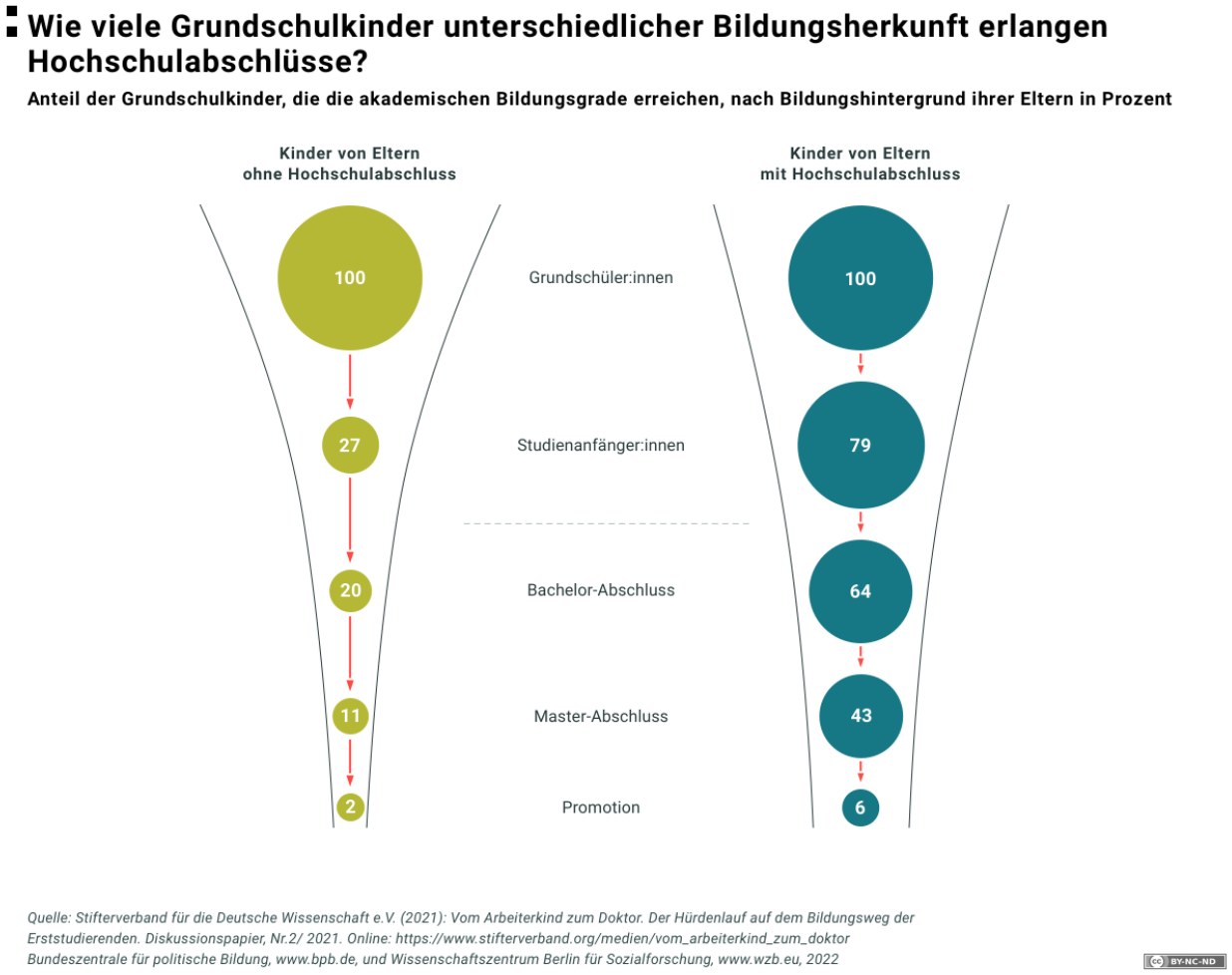


Figure 13.2: We do not all go to University under the same conditions. Everyone needs a support system.

13.5 Tools

- Meditation.
  - Various guided meditations in the free app *insight timer* `https://insighttimer.com/`.
  - Free meditations in the *Plum Village app* `https://plumvillage.app/`.
- Breathing techniques.
- Journaling.
- End-of-week (or day) reflection.
- Remembrances.
- Reading books for personal growth (there is a section at our Faculty's library).
- The Work, by Katie Byron.
- Many other resources and approaches like, Cognitive Behavioral Therapy (CBT).

Find what works for you.

Chapter 14

Managing time

14.1 The weekly planer

How to make a weekly planner:

1. Put all your classes and fixed appointments.
2. Split your independent study time between courses and also between theory and exercises.
3. Aim at having around 40 hours of work a week.
4. Keep adjusting according to what works best for you.

I recommend using a digital calendar.

14.2 How to stick to a study plan

Some things that you should take into account:

- There is an initial barrier (an initial resistance) every time we change or start an activity.
- It takes time to get used to working 40 hours a week, just like it takes time to get used to going to the gym. Even after a short break, resuming work can feel difficult.
- Study with complete focus and involvement. Do not allow hesitation, self-criticism, self-doubt, boredom to hinder your work. Remember Star Wars: "Do it, or don't, there is no try". So if you decide to study, do it with full involvement.
- Discipline can get you only so far. To make an activity sustainable in the long run, you must enjoy it. So learn to do things joyfully or at least without so much resistance.
- Things may not be easy, but the way forward is to overcome limitations every day.

Some concrete tips You created a study program for yourself. Here are a few tips to help you stick to your plan. This is highly individual, so see what works for you.

- **Work with others.** Keep each other accountable.
- **Go to a place where you cannot talk or do anything else apart from studying** (like a library).

- **Ditch your phone** (and computer or tablet if they are distracting you).
- At home: Create a separation between study time and space, and everything else (e.g., use a study lamp).
- **Use a system like the Pomodoro technique**, if that works for you.
- **Stay hydrated**.
- **Keep your materials organized.** Make sure you can easily access your lecture and exercise notes.
- **Set up achievable goals.**
- **Deep practice**: Keep a diary of your study habits to understand what works and what doesn't. Include your emotions in this diary.
- **Learn to deal with frustration**; it only leads to procrastination.
- **Have other activities** (especially physical exercise or artistic hobbies).
- **Adopt a revision-oriented attitude**: if it helps you, imagine that the exam is next week and study under this mindset. Review material before and after each class.
- **Find ways to measure progress.**
- **Study in a structured way**, see Chapter 2.

14.3 Saving time/ Wasting time

In the summer semester of 2025, I asked students what helps them save time and what makes them waste time. Here are their answers.

Saving time.

- Self-created pressure by promising results.
- Asking and answering questions.
- Have a set of well-organised and clean notes.
- Tutorials.
- Meditation or exercises to sharpen the mind.
- Taking breaks.
- Clear objectives for a study session.
- No interruptions, no constant switching.
- Right study environment.
- Wake up early and study at my own pace.
- Prepare the theory class

Wasting time (or making me go slower).

- Worrying with no action.
- Social media.
- Wrong priorities.
- Beautifying notes.
- Reading distressing news.
- Looking online for answers or easier explanations.
- Adopting the study rhythm of others.
- Not revising the theory class but re-watching the recordings of the class.
- Having wrong expectations.

Chapter 15

Studying mathematics

Listen to this video by David Hubermann on how successful students approach their learning.

https://youtu.be/jMhhaAQK1NQ?si=REbCKLrv9TXkypfW

Exercise. Testing oneself is key. How can you do this when studying mathematics?

Revise chapter 5.

Chapter 16

Preparing for an exam

1. **Gather your study materials** Ensure you have all the necessary materials ready, including lecture notes, your personal notes, exercises, and any other resources you've been using throughout the semester.
2. **Make a topic list** Create a comprehensive list of all the topics that need to be covered. Based on your experience during the semester, estimate how much time you'll need to revise each topic.
3. **Organise your study schedule** Plan your time effectively by creating a schedule that covers all the topics on your list. Allocate specific time slots for each subject or section leading up to the exam day.
4. **Build in flexibility** Set aside some extra time as a buffer in case certain topics take longer than expected or unforeseen circumstances arise. This will help create a more resilient plan.
5. **Plan for a final review** Reserve time for a final overall revision to consolidate your understanding and address any remaining weak areas.
6. **Adapt your plan when needed** Be prepared to reassess and adjust your plan as circumstances change. Things rarely go perfectly according to plan, so flexibility is key.
7. **Track your progress** Cross items off your plan as you complete them. Not only does this help you stay organized, but it also provides a sense of accomplishment.
8. **Cover all topics** Aim to review all the material rather than focusing deeply on a few topics while neglecting others. A broad understanding of everything is more effective than selective expertise.
9. **Cover both theory and exercises** Revise both theoretical concepts and exercises. For exercises, consider a mix of approaches:
    - Re-do some exercises by explicitly writing out the answers to reinforce your understanding.
    - Solve others mentally to save time and increase speed.
10. **Stick to your plan** Stay disciplined and follow your study schedule. Consistency is key to thorough preparation.
11. **Remember the fundamentals** Sleep plenty, eat well, do sport, smile, meet people who lift your spirits, and relax from time to time

Chapter 17

Panic: complete blockage

If you are so stress that every time you try to work you get blocked and cannot advance, read chapter 12 ’Panic’ of How to study for a mathematics degree by Lara Alcock (online in u:search) (or in German Wie man erfolgreich Mathematik studiert : Besonderheiten eines nicht-trivialen Studiengangs, Lara Alcock).

Chapter 18

Anxiety in oral presentations

Many people rank public speaking as their top fear, even above death. Remember, even experienced speakers can feel nervous before a presentation. So you are not alone on this. In this video, let's see why blackboard presentations are so challenging and what we can do to overcome it. The video includes read testimonies from senior students.

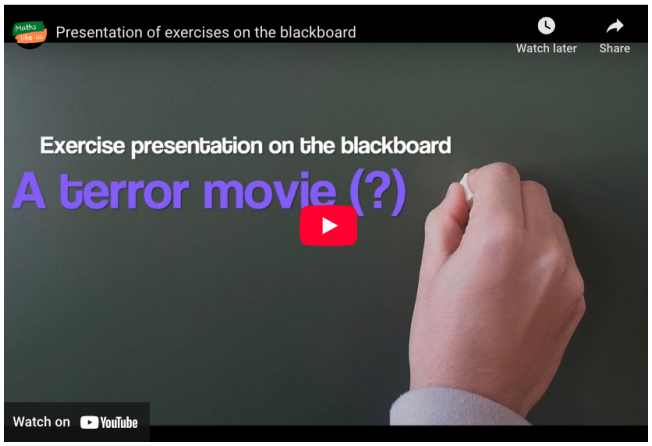


Figure 18.1: https://www.youtube.com/watch?v=xMlLS3-Gq7o

Some pointers to try to control your anxiety.

1. Techniques during the presentation:
    - **Do the double-inhale technique** (breathing exercise) to reduce anxiety: you can do it 2 or 3 times, even just that helps. See the double-inhale technique https://youtu.be/rBdhqBGqiMc?si=TjchgGtkOGOoiYcZ&t=78
    - **Maintain good body posture**: avoid putting tension in your jaw, shoulders, and fists; do not hunch your shoulders, try to have an open chest; you can do a slight smile to relax your face and send a relaxing signal to your brain. It is easier to control your body than your mind, that is why it is good to focus on your body. Relax your body, and the mind will follow.
2. Techniques outside the presentation
    - **Visualization technique**: consciously, visualize in as much detail as possible yourself presenting an exercise, consider multiple scenarios, visualize how people respond and how you respond. Use this to (1) trick your brain into believing that this situation is a common - so that the experience of presenting does not feel so new; (2) to deal with any possible scenario that makes you anxious and prepare yourself for it.

- **What is the worst that can happen?** Ask yourself this question to gain perspective on your anxiety. Often, the worst-case scenario isn't as catastrophic as we imagine. By considering the potential outcomes, you can reduce fear by realizing that even if something goes wrong, it's not the end of the world. This is a stoic tool.
- **What does this say about me?** It is not the presentation perse that scares us, but our perception of it. Particularly, we tend to infer many things about ourselves from it. What does this say about me if my answer is wrong?; if I do not know how to explain myself?; if people laugh at my handwriting?; if my voice sounds weird when I talk? Will people laugh at me? Will the teacher not like me? Will they discover that I cannot do maths? etcetera, etcetera. The mind is the queen of drama. Listen to the drama, examine it, and you will understand, that you do not have to believe it.
- **Give yourself a treat.** Reward yourself for stepping out of your comfort zone and completing the presentation. You did it! Independently of the outcome, you stepped out. Congrats! It is not a small feat.
- **Practice in other contexts and/or record yourself**. Explain your solutions to your colleagues, try to make it realistic. Practice in places where you feel safe. The key is to gain familiarity with presenting and, hopefully, get encouragement from nice colleagues.

Final word:

- **If you try to avoid it now, it will handicap you later.** I know it is daunting, but in many spheres of life you will need to go out there and speak in public. The sooner you get used to it, the better. You will probably always feel a bit stressed and, as remarked above, that is not necessarily a bad thing. But do not let this fear stop you. Do not stop pursuing your goals because of this.

Chapter 19

Experience from senior students

I asked to senior students: What would you have liked to know when you started studying mathematics at University? Here are some answers:

1. **Trust the process.** I had months of struggle, but in the end, things worked out. It takes time to get a feeling for the concepts, and it's okay if you do not understand math during the STEOP.
2. **University math is different from school math.** It took me a while to adapt to studying, as university-level math was a big shift from what I knew in school. Making friends also took time, even though everyone at the faculty of math was super nice. For some, it's also their first time away from home.
3. **Not everyone knows more than you.** The ones that seem the "cleverest" are very visible. The front row always seems the smartest. Being good at school math doesn't necessarily mean you'll excel in university math, and people sometimes act smarter than they really are.
4. **Differences in entry math levels even out over time.** Although there are differences in math backgrounds, things eventually level out. However, it's natural to compare yourself to others.
5. **Going to the blackboard can be intimidating.** Many people find this challenging at first, but it's something you get used to over time. Hearing others' experiences can be reassuring.
6. **Exam periods can be stressful.** Unlike in school, where evaluation is continuous, university exams felt like an "all or nothing" situation, which is new for many students. Remember, you don't have to take all the exams at once or sit for the first available date.
7. **Fear of failure is common.** Many students are afraid of failing and might avoid taking exams even if they are well-prepared.
8. **Study with others**[1]**.** At first, I did the exercises alone, but as the material got harder, I started to fall behind. It wasn't until then that I looked for study partners. Learning in groups is not cheating!
9. **Make use of tutorials and ask questions.** Have the courage to ask questions, there are sure others who don't understand it either.
10. **You don't have to be a genius to understand maths.**

[1] Some people prefer to work alone, and that is absolutely fine, just see what works best for you

Chapter 20

Joy: the only success

> You are studying mathematics because you think that (at one level or another) it will bring you happiness. So the ultimate success is to do mathematics joyfully.

Think about the sentence above, is that true for you?

We start studying mathematics because we think it is going to make us happy, but maths is a very challenging subject, and with it, many times comes suffering. But suffering does not come from math itself but from the meaning and identity we attach to it.

> "People are disturbed not by things, but by the views which they take of things." - Epictetus (a stoic philosopher)

It is a widespread: people keep suffering from the feeling of not being "good enough", or of not belonging to where they want to be; or feeling that they have to prove themselves because their personal worth is not guaranteed.

Self-worth. Self-confidence. Personal meaning. Personal identity. All this is going to be strongly challenged when you study maths.

> "Tension is who you think you should be, relaxation is who you are" - Chinese proverb

When you are doing maths, managing all this psychological hurdle is more difficult than maths itself. But managing ourselves is the art of life (and way beyond the scope of this course). So I invite you to use the psychological challenge that maths brings to grow as a person and flourish in all aspects of live. For this you can get support on the wisdom of previous generations in the form of philosophy and powerful practices like meditation (mindfulness), journal writing, and breathing techniques.

Below there is a small snapshot of some techniques and books to guide you further in the art of self-management, which is the art of life.

20.1 Questions for self-reflection

Questions to choose your action in difficult situations:

1. What is under my control? What is not under my control? (stoic principles)
2. Do my actions/point of view/way of thinking benefit me or harm me?

3. What is fact and what is interpretation/narrative?
4. Does this get me to where I want to go or further away from it?

Questions to understand how situations are affecting us:

1. What am I unwilling to feel? (discover aversions/resistance)
2. What are my deepest intentions?
3. What does the situation say about me? (or what do I think that the situation says about me?).
4. Will I care about this in a week?