Exponential domination and its bidual in function spaces

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Introduction

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If A and B are subsets of a space X, we say that A *dominates* B if $B \subset \overline{A}$. The set B is said to be κ-*dominated*, where κ is a cardinal, if there is a set of size $\leqslant \kappa$ that dominates B. If a space X has density not exceeding κ, then there is a set $A \subset X$ with $|A| \leqslant \kappa$ such that $\overline{A} = X$ and, in particular, A dominates all subsets of X.

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Since it is not always easy to see whether there is a small set in a space X that dominates all other sets, it is natural to reduce the respective quest to finding out whether small subsets of X dominate some other class of small subsets of X. It turns out that κ-domination of some small subsets is equivalent to having density not exceeding κ.

Theorem 1.

Assume that $\kappa \geqslant \omega$ is a cardinal and X is a regular space. Then $d(X) \leqslant \kappa$ if and only if every subset of X of cardinality $\leqslant (2^\kappa)^+$ is contained in the closure of a set of power $\leqslant \kappa$.

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Proof.

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Proof.

We must only prove sufficiency so assume that every $A \in [X]^{\leqslant (2^\kappa)^+}$ is contained in the closure of a set of power $\leqslant \kappa$.

If $d(X) > 2^\kappa$, then we can find a left-separated subspace $D \subset X$ with $|D| = (2^\kappa)^+$. There exists a set $A \subset X$ such that $|A| \leqslant \kappa$ and $D \subset \overline{A}$.

Since X is regular, we have the inequalities $hd(\overline{A}) \leqslant w(\overline{A}) \leqslant 2^\kappa$. On the other hand, $hd(\overline{A}) \geqslant |D| > 2^\kappa$; this contradiction shows that there exists a dense set $E \subset X$ with $|E| \leqslant 2^\kappa$. By our hypothesis, we can find a set $B \subset X$ such that $|B| \leqslant \kappa$ and $E \subset \overline{B}$. It is immediate that B is dense in X so $d(X) \leqslant \kappa$.

Therefore it is a very natural question to ask what happens if every subset of power $\leqslant 2^\kappa$ in a space X is κ-dominated.

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Definition 2.

Given an infinite cardinal κ, we say that X is a space with exponential κ-domination if, for every set $A \subset X$ with $|A| \leqslant 2^\kappa$ there exists $B \subset X$ such that $|B| \leqslant \kappa$ and $A \subset \overline{B}$.

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We will see that the class of spaces with exponential κ-domination has nice categorical properties.

Proposition 3.

Let X be a space with exponential κ-domination. Then

(a) if X is regular, then every subspace of X is 2^κ-monolithic;

(b) every open subset of X has exponential κ-domination;

(c) every continuous image of X features exponential κ-domination;

(d) $d(X) \leqslant \kappa$ if and only if $d(X) \leqslant 2^\kappa$.

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Proposition 4.

Suppose that X is a space and $Y_\alpha \subset X$ features exponential κ-domination for every $\alpha < \kappa$. Then the set $Y = \bigcup_{\alpha<\kappa} Y_\alpha$ also has exponential κ-domination.

Theorem 5.

Assume that X_t is a space with exponential κ-domination for every $t \in T$ and a point u is chosen in the product $X = \prod\{X_t : t \in T\}$. Then the Σ_{2^κ}-product $\Sigma_{2^\kappa}(X, u) = \{x \in X : |\{t \in T : x(t) \neq u(t)\}| \leqslant 2^\kappa\}$ also features exponential κ-domination.

Theorem 5.

Assume that X_t is a space with exponential κ-domination for every $t \in T$ and a point u is chosen in the product $X = \prod\{X_t : t \in T\}$. Then the Σ_{2^κ}-product $\Sigma_{2^\kappa}(X, u) = \{x \in X : |\{t \in T : x(t) \neq u(t)\}| \leqslant 2^\kappa\}$ also features exponential κ-domination.

Proof.

For any set $S \subset T$ consider the projection $p_S : X \to X_S = \prod_{t \in S} X_t$ and let $u_S = p_S(u)$. Given any point $x \in \Sigma_{2^\kappa}(X, u)$, the cardinality of the set $\mathrm{supp}(x) = \{t \in T : x(t) \neq u(t)\}$ does not exceed 2^κ.

Take any set $A \subset \Sigma_{2^\kappa}(X, u)$ with $|A| \leqslant 2^\kappa$. Then the cardinality of the set $S = \bigcup\{\mathrm{supp}(x) : x \in A\}$ does not exceed 2^κ and A is contained in the set $Y = X_S \times \{u_{T \setminus S}\}$ which is, evidently, homeomorphic to X_S. Since $d(X_S) \leqslant \kappa$, any dense subset of Y of cardinality κ witnesses exponential κ-domination of the space $\Sigma_{2^\kappa}(X, u)$.

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Corollary 6.

If X_t is a space with exponential κ-domination for all $t \in T$ and $|T| \leqslant 2^\kappa$, then the product $X = \prod_{t \in T} X_t$ also has exponential κ-domination.

Theorem 5 makes it possible to construct non-separable spaces featuring exponential ω-domination.

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Example 7.

Given a regular infinite cardinal $\kappa > \mathfrak{c}$, consider the $\Sigma_{\mathfrak{c}}$-product $E_\kappa = \{x \in \mathbb{D}^\kappa : |\{\alpha : x(\alpha) \neq 0\}| \leqslant \mathfrak{c}\}$ in the Cantor cube $\mathbb{D}^\kappa$. It follows from Theorem 5 that E_κ is a space with exponential ω-domination. It is standard to see that E_κ is countably compact and $d(E_\kappa) = \kappa$. In particular, exponential ω-domination of a countably compact space does not imply its separability. If $u(\alpha) = 0$ for each $\alpha < \kappa$, then $u \in E_\kappa$ and the space $E_\kappa \backslash \{u\}$ also features exponential ω-domination by Proposition 3(b). We omit an easy proof of the fact that $ext(E_\kappa \backslash \{u\}) = \kappa$.

Example 8.

Let $\kappa > \mathfrak{c}$ be an infinite regular cardinal and consider again the $\Sigma_{\mathfrak{c}}$-product E_κ in the Cantor cube $\mathbb{D}^\kappa$. Letting $p(\alpha) = 1$ for all $\alpha < \kappa$ we define a point $p \in \mathbb{D}^\kappa$. If $X = E_\kappa \cup \{p\}$, then X is a space with exponential ω-domination by Theorem 5 and Proposition 4. It is standard to see that $p \notin \overline{A}$ whenever $A \subset E_\kappa$ and $|A| < \kappa$; this shows that $t(X) = \kappa$. Thus, the tightness of a space with exponential ω-domination can be arbitrarily large.

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Proposition 9.

Suppose that X_t is a space with $|X_t| > 1$ for every $t \in T$. If $|T| > 2^\kappa$, then the product $X = \prod_{t \in T} X_t$ has no exponential κ-domination.

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Proposition 9.

Suppose that X_t is a space with $|X_t| > 1$ for every $t \in T$. If $|T| > 2^\kappa$, then the product $X = \prod_{t\in T} X_t$ has no exponential κ-domination.

Proof.

If $|T| > 2^\kappa$, then the Cantor cube $K = \mathbb{D}^{(2^\kappa)^+}$ embeds in X. The density of K does not exceed 2^κ; this, together with $w(K) > 2^\kappa$ shows that K is not 2^κ-monolithic which is a contradiction with Proposition 3(a).

Proposition 10.

Given an infinite cardinal κ, if a space X has exponential κ-domination, then κ^+ is a caliber of X. In particular, $c(X) \leqslant \kappa$.

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Proof.

Assume that $\mathcal{U} \subset \tau^*(X)$ is a family of cardinality κ^+ and choose a point $x_U \in U$ for every $U \in \mathcal{U}$. The set $A = \{x_U : U \in \mathcal{U}\}$ has cardinality not exceeding 2^κ so there exists a set $B \subset X$ with $|B| \leqslant \kappa$ such that $A \subset \overline{B}$. Then $U \cap B \neq \emptyset$ for every $U \in \mathcal{U}$ so $\mathcal{U} = \bigcup\{\mathcal{U}_b : b \in B\}$ where $\mathcal{U}_b = \{U \in \mathcal{U} : b \in U\}$ for each $b \in B$. By regularity of κ^+, we must have $|\mathcal{U}_b| = \kappa^+$ for some $b \in B$ so $\mathcal{U}_b$ witnesses that κ^+ is a caliber of X.

Proposition 11.

If X is a κ-monolithic space with exponential κ-domination, then $nw(X) \leqslant \kappa$.

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Proof.

If $Y \subset X$ is a right-separated subspace and $|Y| = \kappa^+$, then there exists a set $A \subset X$ with $|A| \leqslant \kappa$ that dominates Y. It follows from the equalities $hl(\overline{A}) \leqslant nw(\overline{A}) \leqslant \kappa$ that $\overline{A}$ cannot have right-separated subsets of cardinality κ^+. This contradiction shows that $hl(X) \leqslant \kappa$ and therefore $d(X) \leqslant |X| \leqslant 2^\kappa$ so $d(X) \leqslant \kappa$ by Proposition 3(d). Now it follows from κ-monolithity of X that $nw(X) \leqslant \kappa$.

Proposition 12.

Let X be a space with exponential κ-domination such that $s(X) \leqslant \kappa$. Then $d(X) \leqslant \kappa$.

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Proof.

There exists a set $Y \subset X$ such that $\overline{Y} = X$ and $hl(Y) \leqslant \kappa$. Therefore we have the inequalities $d(X) \leqslant |Y| \leqslant 2^{hl(Y)} \leqslant 2^{\kappa}$ so $d(X) \leqslant \kappa$ by Proposition 3(d).

Corollary 13.

If X is a hereditarily collectionwise Hausdorff space with exponential κ-domination, then $d(X) \leqslant \kappa$.

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Proof.

If $s(X) > \kappa$, then fix a discrete subspace $D \subset X$ with $|D| > \kappa$ and let $E = \overline{D} \backslash D$; then D is a closed discrete subspace of $Y = X \backslash E$. By collectionwise Hausdorffness of Y there exists a disjoint family $\mathcal{U} = \{U_d : d \in D\}$ of open subsets of Y such that $d \in U_d$ for each $d \in D$. It immediate that $\mathcal{U} \subset \tau^*(X)$ and $|\mathcal{U}| > \kappa$; this contradiction with $c(X) \leqslant \kappa$ (see Proposition 10) shows that $s(X) \leqslant \kappa$ and hence $d(X) \leqslant \kappa$ by Proposition 12.

Corollary 14.

If X is a monotonically normal space featuring exponential κ-domination, then $d(X) \leqslant \kappa$. In particular, any generalized ordered space X with exponential κ-domination has density not exceeding κ.

Example 15.

Given a cardinal $\mu > \mathfrak{c}$, let $Y = \{x \in \mathbb{R}^\mu : |x^{-1}(\mathbb{R}\backslash\{0\})| \leqslant \omega\}$ be the Σ-product in the space $\mathbb{R}^\mu$. If $X = \{x \in \mathbb{R}^\mu : |x^{-1}(\mathbb{R}\backslash\{0\})| \leqslant \mathfrak{c}\}$ is the $\Sigma_{\mathfrak{c}}$-product in $\mathbb{R}^\mu$, then X has exponential ω-domination by Theorem 5. The set Y is dense in X; since it is ω-monolithic and non-separable, it does not have exponential ω-domination (see Proposition 11). Therefore a dense subspace of a space with exponential ω-domination need not feature exponential ω-domination.

Example 8 shows that, for any cardinal λ, a space X with exponential ω-domination can have a point p which is not in the closure of any subset of $X \backslash \{p\}$ of cardinality less than λ. The following proposition shows that a space with exponential κ-domination cannot have many such points. Given a cardinal μ, we say that a point x of a space X is *μ-accessible* from a set $A \subset X$ if $x \in \overline{B}$ for some $B \subset A$ with $|B| \leqslant \mu$.

Proposition 16.

If X is a space with exponential κ-domination, then there is a set $A \subset X$ such that $|A| \leqslant \kappa$ and every point $x \in X \backslash A$ is κ-accessible from $X \backslash \{x\}$.

Proposition 16.

If X is a space with exponential κ-domination, then there is a set $A \subset X$ such that $|A| \leqslant \kappa$ and every point $x \in X \setminus A$ is κ-accessible from $X \setminus \{x\}$.

Proof.

If our statement is not true, then there is a set $Y \subset X$ such that $|Y| = \kappa^+$ and no point $y \in Y$ is κ-accessible from $X \setminus \{y\}$. Pick a set $B \subset X$ such that $|B| \leqslant \kappa$ and $Y \subset \overline{B}$. Then the set B witnesses that every point y from a non-empty set $Y \setminus B$ is κ-accessible from $X \setminus \{y\}$ which is a contradiction.

It is easy to infer from Example 7 and Proposition 9 that having a dense subspace with exponential κ-domination does not imply exponential κ-domination. Example 15 shows that a dense subspace of a space with exponential ω-domination need not have exponential ω-domination. This picture changes if we have certain kind of accessibility from a dense subspace.

Proposition 17.

Given a space X and an infinite cardinal κ, assume that Y is a dense subspace of X with exponential κ-domination. If every point of X is 2^κ-accessible from Y, then the space X has exponential κ-domination.

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Proof.

Given any $A \in [X]^{\leqslant 2^\kappa}$, fix a set $Q_a \subset Y$ such that $|Q_a| \leqslant 2^\kappa$ and $a \in \overline{Q_a}$ for every $a \in A$. Observe that the set $Q = \bigcup\{Q_a : a \in A\} \subset Y$ has cardinality not exceeding 2^κ so there exists $B \in [Y]^{\leqslant\kappa}$ with $Q \subset \overline{B}$. Clearly, $A \subset \overline{B}$, i.e., B witnesses exponential κ-domination in X.

Corollary 18.

Let κ be an infinite cardinal and assume that Y is a dense subspace of a space X. If every point of X is κ-accessible from Y, then X features exponential κ-domination if and only if so does Y.

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Proof.

Observe that sufficiency is an immediate consequence of Proposition 17 and assume that X has exponential κ-domination. If $A \in [Y]^{\leqslant 2^\kappa}$, then there exists $B \subset X$ such that $|B| \leqslant \kappa$ and $A \subset \overline{B}$. For each $b \in B$ pick a set $Q_b \subset Y$ such that $b \in \overline{Q_b}$ and $|Q_b| \leqslant \kappa$. Then the set $Q = \bigcup\{Q_b : b \in B\} \subset Y$ has cardinality $\leqslant \kappa$ and $A \subset \overline{Q}$.

Corollary 19.

Suppose that X is a space with $t(X) \leqslant \kappa$ and Y is dense in X. Then X features exponential κ-domination of and only if so does Y.

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Corollary 20.

Suppose that X is a space with $t(X) \leqslant 2^\kappa$ and Y is a dense subspace of X with exponential κ-domination. Then X also features exponential κ-domination.

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Proposition 21.

Suppose that X is a regular space that features exponential κ-domination and $\pi\chi(Y) \leqslant 2^\kappa$ for some dense set $Y \subset X$. Then $d(X) \leqslant \kappa$.

The following statement is the main tool in proving that, in Čech-complete spaces, exponential κ-domination implies that the density does not exceed κ.

Proposition 21.

Suppose that X is a regular space that features exponential κ-domination and $\pi\chi(Y) \leqslant 2^\kappa$ for some dense set $Y \subset X$. Then $d(X) \leqslant \kappa$.

Proof.

It follows from Proposition 10 that $c(X) \leqslant \kappa$ and hence $c(Y) \leqslant \kappa$. By Shapirovsky's theorem, we have the inequalities $d(Y) \leqslant \pi\chi(Y)^{c(Y)} \leqslant (2^\kappa)^\kappa = 2^\kappa$; therefore $d(X) \leqslant d(Y) \leqslant 2^\kappa$ which implies $d(X) \leqslant \kappa$ by Proposition 3(d).

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Lemma 22.

Suppose that λ is an infinite cardinal and F is a closed subspace of a regular space X. If $\chi(F,X) \leqslant \lambda$ and $K \subset F$ is a compact set such that $\chi(K,F) \leqslant \lambda$, then $\chi(K,X) \leqslant \lambda$.

Lemma 23.

Given a Hausdorff space X with exponential κ-domination and a compact subspace $K \subset X$, there exists no continuous map of K onto the Tychonoff cube $\mathbb{I}^{(2^\kappa)^+}$.

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Proof.

Suppose that $f : K \to \mathbb{I}^{(2^\kappa)^+}$ is a continuous onto map and fix a set $E \subset K$ such that $|E| \leqslant 2^\kappa$ and $f(E)$ is dense in $\mathbb{I}^{(2^\kappa)^+}$. There exists a set $D \subset X$ such that $|D| \leqslant \kappa$ and $E \subset \overline{D}$. Given any $\alpha < \lambda = (2^\kappa)^+$, the sets $A_\alpha = \{x \in K : f(x)(\alpha) \in [0, \frac{1}{3}]\}$ and $B_\alpha = \{x \in K : f(x)(\alpha) \in [\frac{2}{3}, 1]\}$ are compact and disjoint so we can find disjoint open sets U_α and V_α in the space X such that $A_\alpha \subset U_\alpha$ and $B_\alpha \subset V_\alpha$.

If $\alpha < \beta < \lambda$, then there exists a point $z \in E$ such that $f(z)(\alpha) \in (0, \frac{1}{3})$ and $f(z)(\beta) \in (\frac{2}{3}, 1)$ whence $z \in A_\alpha \cap B_\beta \subset U_\alpha \cap V_\beta$. Since $z \in \overline{D}$, there exists a point $d \in D$ such that $d \in U_\alpha \cap V_\beta$ and therefore $d \in U_\alpha \backslash U_\beta$. This proves that the map $\alpha \to U_\alpha \cap D$ is an injection from λ to the power set of D, i.e., the set D of cardinality $\leqslant \kappa$ has at least $\lambda = (2^\kappa)^+$-many distinct subsets which is a contradiction.

Theorem 24.

Assume that X is a regular space with exponential κ-domination and there exists a set $Y \subset X$ such that $\overline{Y} = X$ and every $y \in Y$ belongs to a compact set $K_y \subset X$ with $\chi(K_y, X) \leqslant 2^\kappa$. Then $d(X) \leqslant \kappa$.

Theorem 24.

Assume that X is a regular space with exponential κ-domination and there exists a set $Y \subset X$ such that $\overline{Y} = X$ and every $y \in Y$ belongs to a compact set $K_y \subset X$ with $\chi(K_y, X) \leqslant 2^\kappa$. Then $d(X) \leqslant \kappa$.

Proof.

Take any non-empty open set $U \subset X$ and fix a point $y \in U \cap Y$. Using the compact set K_y and Lemma 22, it is easy to find a compact set $L \subset U$ such that $y \in L$ and $\chi(L, X) \leqslant 2^\kappa$. If the set $\{x \in L : \pi\chi(x, L) \leqslant 2^\kappa\}$ is empty, then there exists a surjective continuous map $f : L \to \mathbb{I}^{(2^\kappa)^+}$ of L onto the Tychonoff cube $\mathbb{I}^{(2^\kappa)^+}$ which contradicts Lemma 23.

Therefore there exists a point $x \in L$ such that $\pi\chi(x, L) \leqslant 2^\kappa$. Take a π-base $\mathcal{B}$ at the point x in L such that $|\mathcal{B}| \leqslant 2^\kappa$ and choose, for every $B \in \mathcal{B}$, a non-empty compact set $P_B \subset B$ which is a G_δ-subset of L and hence $\chi(P_B, L) \leqslant \omega$. Applying Lemma 22 again we conclude that $\chi(P_B, X) \leqslant 2^\kappa$ for every $B \in \mathcal{B}$. If $\mathcal{U}_B$ is an outer base of P_B in X with $|\mathcal{P}_B| \leqslant 2^\kappa$, then $\mathcal{U} = \bigcup\{\mathcal{U}_B : B \in \mathcal{B}\}$ is a local π-base of X at x and $|\mathcal{U}| \leqslant 2^\kappa$.

Therefore there exists a point $x \in L$ such that $\pi\chi(x, L) \leqslant 2^\kappa$. Take a π-base $\mathcal{B}$ at the point x in L such that $|\mathcal{B}| \leqslant 2^\kappa$ and choose, for every $B \in \mathcal{B}$, a non-empty compact set $P_B \subset B$ which is a G_δ-subset of L and hence $\chi(P_B, L) \leqslant \omega$. Applying Lemma 22 again we conclude that $\chi(P_B, X) \leqslant 2^\kappa$ for every $B \in \mathcal{B}$. If $\mathcal{U}_B$ is an outer base of P_B in X with $|\mathcal{P}_B| \leqslant 2^\kappa$, then $\mathcal{U} = \bigcup\{\mathcal{U}_B : B \in \mathcal{B}\}$ is a local π-base of X at x and $|\mathcal{U}| \leqslant 2^\kappa$.

Since we found, in every non-empty open subset of the space X, a point of local π-character not exceeding 2^κ, the set $Z = \{x \in X : \pi\chi(x, X) \leqslant 2^\kappa\}$ is dense in the space X and, evidently, $\pi\chi(Z) \leqslant 2^\kappa$. Finally, apply Proposition 21 to conclude that $d(X) \leqslant \kappa$.

Corollary 25.

If a space X features exponential κ-domination and has a dense Čech-complete subspace, then $d(X) \leqslant \kappa$.

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Proof.

If Y is a dense Čech-complete subspace of X, then Y is of point-countable type, i.e., for every $y \in Y$, there exists a compact set $K_y \subset Y$ with $y \in K_y$ and $\chi(K_y, Y) \leqslant \omega$. Then $\chi(K_y, X) \leqslant \omega$ because Y is dense in X so we can apply Theorem 24 to see that $d(X) \leqslant \kappa$.

Corollary 26.

Suppose that a space X features exponential κ-domination and the set G of points of local compactness of X is dense in X. Then $d(X) \leqslant \kappa$.

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Corollary 27.

A Čech-complete space X is separable if and only if every subset $A \subset X$ with $|A| \leqslant \mathfrak{c}$ is contained in the closure of a countable set.

Example 28.

Define a point $u \in \mathbb{I}^{\mathfrak{c}^+}$ by $u(\alpha) = 0$ for any $\alpha < \mathfrak{c}^+$. Then there exists a subspace $X \subset \Sigma_{\mathfrak{c}}(\mathbb{I}^{\mathfrak{c}^+}, u)$ with the following properties:

(a) X is dense in $\mathbb{I}^{\mathfrak{c}^+}$;

(b) $|X| = \mathfrak{c}^+$ and $\psi(X) \leqslant \omega$;

(c) X is a non-separable space with exponential ω-domination.

In particular, a Tychonoff space of countable pseudocharacter with exponential ω-domination need not be separable or have cardinality $\leqslant \mathfrak{c}$; this should be compared with Proposition 21.

Proof.

Given any $S \subset \mathfrak{c}^+$, we will need the projection $p_S : \mathbb{I}^{\mathfrak{c}} \to \mathbb{I}^S$ and the point $u_S = p_S(u) \in \mathbb{I}^S$. Denote by N the sequence $\{2^{-n} : n \in \omega\} \subset \mathbb{I}$. It is clear that N converges to 0; let $M = \mathbb{I} \setminus N$. Consider any ordinal $\alpha < \mathfrak{c}^+$; it is easy to find a countable dense set A_α in $M^\alpha \times \{u_{\mathfrak{c}^+ \setminus \alpha}\}$.

Choose a faithful enumeration $\{a_n^\alpha : n \in \omega\}$ of the set A_α; letting $x_n^\alpha(\beta) = a_n^\alpha(\beta)$ for any $\beta \neq \alpha$ and $x_n^\alpha(\alpha) = 2^{-n}$ for every $n \in \omega$, we obtain a countable set

$$X_\alpha = \{x_n^\alpha : n \in \omega\} \subset \mathbb{I}^{\alpha+1} \times \{u_{\mathfrak{c}^+ \setminus (\alpha+1)}\}$$

such that $I_\alpha = \mathbb{I}^\alpha \times \{u_{\mathfrak{c}^+ \setminus \alpha}\} \subset \overline{X}_\alpha$. The family $\{X_\alpha : \alpha < \mathfrak{c}^+\}$ is disjoint because

$$(1) \qquad p_{\{\alpha\}}(X_\beta) \cap p_{\{\alpha\}}(X_\alpha) = \emptyset \text{ whenever } \alpha \neq \beta.$$

Therefore the set $X = \bigcup\{X_\alpha : \alpha < \mathfrak{c}^+\}$ has cardinality $\mathfrak{c}^+$. The equality (1) easily implies that X has countable pseudocharacter. The fact that $I_\alpha \subset \overline{X}_\alpha$ for every $\alpha < \mathfrak{c}^+$ easily implies that X is dense in $\mathbb{I}^{\mathfrak{c}^+}$ and hence we proved (a) and (b).

If $A \subset X$ is an arbitrary set with $|A| \leqslant \mathfrak{c}$, there exists an ordinal $\alpha < \mathfrak{c}^+$ such that $A \subset \bigcup\{X_\beta : \beta < \alpha\} \subset I_{\alpha+1} \subset \overline{X}_{\alpha+1}$ and hence the set $X_{\alpha+1}$ witnesses exponential ω-domination in X. A similar reasoning shows that X is not separable but, instead of proving everything from scratch, we prefer to observe that if X is separable, then $w(X) \leqslant \mathfrak{c}$ and therefore

$$\mathfrak{c}^+ = |X| \leqslant nw(X)^{\psi(X)} \leqslant w(X)^{\psi(X)} \leqslant \mathfrak{c}^\omega = \mathfrak{c};$$

this contradiction shows that (c) holds as promised.

1. Question.

Suppose that X is a σ-compact space that features exponential ω-domination. Must X be separable?

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3. Question.

Suppose that X is a Lindelöf space with exponential ω-domination. Must X be separable?

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Suppose that X is a σ-compact space that features exponential ω-domination. Must X be separable?

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Suppose that X is a space with a G_δ-diagonal that features exponential ω-domination. Must X be separable?

3. Question.

Suppose that X is a Lindelöf space with exponential ω-domination. Must X be separable?

4. Question.

Suppose that X is a Lindelöf space with exponential ω-domination. Is it true that $|X| \leqslant 2^{\mathfrak{c}}$?

From this moment on, all spaces under consideration are assumed to be Tychonoff. Recall that $iw(X) = \min\{w(Y) :$ the space Y is an image of X under a continuous bijection$\}$. The cardinal $iw(X)$ is called the i-weight of X. The i-weight is bidual to density with respect to C_p-functor, that is, $d(X) = iw(C_p(X))$ and $iw(X) = d(C_p(X))$ for any space X. Not so many cardinal invariants have a bidual under the C_p-functor. A classical example is the network weight because $nw(X) = nw(C_p(X))$, i.e., nw is bidual to itself. Since exponential κ-domination is a generalization of density, it is natural to suspect that it has a bidual cardinal invariant. We will show that this is, indeed, the case.

Definition 29.

Given an infinite cardinal κ, we will say that a space X is *exponentially κ-cofinal* if for any continuous onto map $f : X \to Y$ such that $w(Y) \leqslant 2^\kappa$, there exist continuous surjective maps $g : X \to Z$ and $h : Z \to Y$ such that $iw(Z) \leqslant \kappa$ and $h \circ g = f$.

The proof of the following proposition is straightforward from the definition. It shows that exponential κ-cofinality is a weakening of the property of having *i*-weight not exceeding κ. Therefore an important line of study of exponential κ-cofinality is to find nice classes of spaces in which it coincides with $iw \leqslant \kappa$.

The proof of the following proposition is straightforward from the definition. It shows that exponential κ-cofinality is a weakening of the property of having *i*-weight not exceeding κ. Therefore an important line of study of exponential κ-cofinality is to find nice classes of spaces in which it coincides with $iw \leqslant \kappa$.

Proposition 30.

If $iw(X) \leqslant \kappa$, then X is an exponentially κ-cofinal space. In particular, any space X with $nw(X) \leqslant \kappa$ is exponentially κ-cofinal.

Proposition 31.

Assume that X is an exponentially κ-cofinal space and Y is a continuous image of X with $iw(Y) \leqslant 2^\kappa$. Then $|Y| \leqslant 2^\kappa$. In particular, if $nw(Y) \leqslant 2^\kappa$, then $|Y| \leqslant 2^\kappa$.

Proposition 31.

Assume that X is an exponentially κ-cofinal space and Y is a continuous image of X with $iw(Y) \leqslant 2^\kappa$. Then $|Y| \leqslant 2^\kappa$. In particular, if $nw(Y) \leqslant 2^\kappa$, then $|Y| \leqslant 2^\kappa$.

Proof.

Let $f : X \to Y$ be a continuous onto map. Take a space M and a condensation $u : Y \to M$ such that $w(M) \leqslant 2^\kappa$ and choose continuous onto maps $v : X \to Z$ and $w : Z \to M$ such that $iw(Z) \leqslant \kappa$ and $w \circ v = u \circ f$. It follows from $|Z| \leqslant 2^{iw(Z)} \leqslant 2^\kappa$ that $|M| \leqslant 2^\kappa$ and hence $|Y| = |M| \leqslant 2^\kappa$.

Corollary 32.

Any discrete exponentially κ-cofinal space has cardinality not exceeding 2^κ.

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Any discrete exponentially κ-cofinal space has cardinality not exceeding 2^κ.

Proof.

If X is a discrete exponentially κ-cofinal space and $|X| > 2^\kappa$, then there exists a surjective map $f : X \to Y \subset \mathbb{I}^{2^\kappa}$ such that $|Y| > 2^\kappa$. Since the map f is continuous and $w(Y) \leqslant 2^\kappa$, we have a contradiction with Proposition 31.

Proposition 33.

If X is an exponentially κ-cofinal space and a set $Y \subset X$ is C^*-embedded in X, then Y is exponentially κ-cofinal. In particular, if X is a normal exponentially κ-cofinal space, then every closed subspace of X is exponentially κ-cofinal.

Proposition 33.

If X is an exponentially κ-cofinal space and a set $Y \subset X$ is C^*-embedded in X, then Y is exponentially κ-cofinal. In particular, if X is a normal exponentially κ-cofinal space, then every closed subspace of X is exponentially κ-cofinal.

Proof.

Take any continuous onto map $f : Y \to Y'$ such that $w(Y') \leqslant \mu = 2^\kappa$. There is no loss of generality to consider that $Y' \subset \mathbb{I}^\mu$ and there is a family $\{f_\alpha : \alpha < \mu\} \subset C_p(X, \mathbb{I})$ such that $f = \Delta\{f_\alpha : \alpha < \mu\}$. Let $u_\alpha : X \to \mathbb{I}$ be a continuous extension of f_α for every $\alpha < \mu$. Then the diagonal product $u = \Delta\{u_\alpha : \alpha < \mu\}$ maps X continuously onto a space $X' \subset \mathbb{I}^\mu$ and $u|Y = f$.

By exponential κ-cofinality of X, we can find continuous onto maps $v : X \to Z$ and $w : Z \to X'$ such that $w \circ v = u$ and $iw(Z) \leqslant \kappa$. Let $g = v|Y$ and $h = w|v(Y)$. Then $g \circ h = f$ and the i-weight of the space $Z' = v(Y)$ does not exceed $iw(Z) \leqslant \kappa$ so the maps g and h witness that Y is exponentially κ-cofinal.

By exponential κ-cofinality of X, we can find continuous onto maps $v : X \to Z$ and $w : Z \to X'$ such that $w \circ v = u$ and $iw(Z) \leqslant \kappa$. Let $g = v|Y$ and $h = w|v(Y)$. Then $g \circ h = f$ and the i-weight of the space $Z' = v(Y)$ does not exceed $iw(Z) \leqslant \kappa$ so the maps g and h witness that Y is exponentially κ-cofinal.

Corollary 34.

Assume that X is an exponentially κ-cofinal space and D is a discrete subset of X. If D is C^*-embedded in X, then $|D| \leqslant 2^\kappa$. In particular, if X is a normal exponentially κ-cofinal space, then $ext(X) \leqslant 2^\kappa$.

Corollary 35.

If κ is an infinite cardinal and X is an exponentially κ-cofinal space, then any discrete family of non-empty open subsets of X has cardinality not exceeding 2^κ.

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If κ is an infinite cardinal and X is an exponentially κ-cofinal space, then any discrete family of non-empty open subsets of X has cardinality not exceeding 2^κ.

Proof.

If $\mathcal{U}$ is a discrete family of non-empty open subsets of the space X and $|\mathcal{U}| > 2^\kappa$, then pick a point $x_U \in U$ for every $U \in \mathcal{U}$. The set $D = \{x_U : U \in \mathcal{U}\}$ is closed, discrete and $|D| = |\mathcal{U}| > 2^\kappa$. Since the set D is C-embedded in X, we have a contradiction with Corollary 34.

Theorem 36.

Given an exponentially κ-cofinal space X and a set $A \subset X$ with $nw(A) \leqslant 2^\kappa$, there exists a continuous map $\varphi : X \to M$ such that $w(M) \leqslant \kappa$ and $\varphi|A$ is an injection. In particular, $iw(A) \leqslant \kappa$ and $|A| \leqslant 2^\kappa$.

Theorem 36.

Given an exponentially κ-cofinal space X and a set $A \subset X$ with $nw(A) \leqslant 2^\kappa$, there exists a continuous map $\varphi : X \to M$ such that $w(M) \leqslant \kappa$ and $\varphi|A$ is an injection. In particular, $iw(A) \leqslant \kappa$ and $|A| \leqslant 2^\kappa$.

Proof.

The restriction mapping $\pi_A : C_p(X) \to C_p(A)$ is continuous and the set $E = \pi_A(C_p(X))$ is dense in $C_p(A)$. Observe that $d(E) \leqslant nw(E) \leqslant nw(C_p(A)) = nw(A) \leqslant 2^\kappa$ so we can find a dense subset $D \subset E$ with $|D| \leqslant 2^\kappa$. The set D is also dense in $C_p(A)$ and hence it separates the points of A. If $h = \Delta(D)$ is the diagonal product of the functions from D, then $h : X \to \mathbb{R}^D$; let $Y = h(X)$.

The map $h : X \to Y$ is continuous and surjective; since $w(Y) \leqslant 2^\kappa$, there exists a space Z and continuous onto maps $p : X \to Z$ and $q : Z \to Y$ such that $q \circ p = h$ and $iw(Z) \leqslant \kappa$. Observe that the map $h|A$ is injective and hence so is $p|A$. There exists a condensation $r : Z \to M$ for some space M such that $w(M) \leqslant \kappa$. It is immediate that the map $\varphi = r \circ p$ is as promised.

Corollary 37.

For any exponentially κ-cofinal space X, if $A \subset X$ and $|A| \leqslant \kappa$, then $|\overline{A}| \leqslant 2^\kappa$ and $iw(\overline{A}) \leqslant \kappa$.

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For any exponentially κ-cofinal space X, if $A \subset X$ and $|A| \leqslant \kappa$, then $|\overline{A}| \leqslant 2^\kappa$ and $iw(\overline{A}) \leqslant \kappa$.

Proof.

Just note that $w(\overline{A}) \leqslant 2^\kappa$ and apply Theorem 36.

Corollary 38.

Assume that X is a metrizable exponentially κ-cofinal space. Then $w(X) \leqslant 2^\kappa$ and hence $iw(X) \leqslant \kappa$.

Corollary 38.

Assume that X is a metrizable exponentially κ-cofinal space. Then $w(X) \leqslant 2^\kappa$ and hence $iw(X) \leqslant \kappa$.

Proof.

It follows from Corollary 34 that $w(X) = ext(X) \leqslant 2^\kappa$ by so we can apply Theorem 36 to conclude that $iw(X) \leqslant \kappa$.

Corollary 39.

If X is an exponentially κ-cofinal space and $s(X) \leqslant \kappa$, then $iw(X) \leqslant \kappa$.

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If X is an exponentially κ-cofinal space and $s(X) \leqslant \kappa$, then $iw(X) \leqslant \kappa$.

Proof.

Observe that $nw(X) \leqslant 2^{s(X)} \leqslant 2^{\kappa}$ and apply Theorem 36.

If $iw(X) \leqslant \kappa$, then the diagonal of X is easily seen to be a G_κ-set. This is not necessarily the case for an exponentially κ-cofinal space X but we still have an important weaker property in X. Recall that the diagonal of X is called κ-small, if for any set $A \subset (X \times X)\backslash\Delta$ with $|A| > \kappa$, there exists a set $B \subset A$ such that $|B| = \kappa^+$ and $\overline{B} \cap \Delta = \emptyset$. Here $\Delta = \{(x, x) : x \in X\}$ is the diagonal of the space X.

If $iw(X) \leqslant \kappa$, then the diagonal of X is easily seen to be a G_κ-set. This is not necessarily the case for an exponentially κ-cofinal space X but we still have an important weaker property in X. Recall that the diagonal of X is called κ-small, if for any set $A \subset (X \times X)\backslash\Delta$ with $|A| > \kappa$, there exists a set $B \subset A$ such that $|B| = \kappa^+$ and $\overline{B} \cap \Delta = \emptyset$. Here $\Delta = \{(x, x) : x \in X\}$ is the diagonal of the space X.

Proposition 40.

If X is an exponentially κ-cofinal space, then the diagonal of X is κ^+-small. In particular, any exponentially ω-cofinal space has a small diagonal.

Proof.

Take any faithfully indexed set $A = \{z_\alpha = (x_\alpha, y_\alpha) : \alpha < \kappa^+\}$ contained in $(X \times X)\backslash\Delta_X$; then the cardinality of the set $Y = \{x_\alpha, y_\alpha : \alpha < \kappa^+\}$ does not exceed $\kappa^+ \leqslant 2^\kappa$ and therefore there exists a continuous map $\varphi : X \to M$ such that $w(M) \leqslant \kappa$ and $\varphi|Y$ is injective (see Theorem 36). This implies that the set $\{(\varphi(x_\alpha), \varphi(y_\alpha)) : \alpha < \kappa^+\}$ is contained in $(M \times M)\backslash\Delta_M$ which, together with $w(M) \leqslant \kappa$, guarantees that there is a set $E \subset \kappa^+$ such that $|E| = \kappa^+$ and the closure of the set $\{(\varphi(x_\alpha), \varphi(y_\alpha)) : \alpha \in E\}$ in $M \times M$ does not meet Δ_M. Then the closure of the set $B = \{z_\alpha : \alpha \in E\} \subset A$ in $X \times X$ does not meet Δ_X and $|B| = \kappa^+$, i.e., the set B witnesses that X has a κ^+-small diagonal.

Proposition 41.

Given an infinite cardinal κ, if X is a P_{2^κ}-space with $ext(X) \leqslant 2^\kappa$, then X is exponentially κ-cofinal.

Proposition 41.

Given an infinite cardinal κ, if X is a P_{2^κ}-space with $ext(X) \leqslant 2^\kappa$, then X is exponentially κ-cofinal.

Proof.

Let $f : X \to Y$ be a continuous onto map such that $w(Y) \leqslant 2^\kappa$. Then $f^{-1}(y)$ is open in X being a G_{2^κ}-set for every $y \in Y$. Therefore the partition $\mathcal{P} = \{f^{-1}(y) : y \in Y\}$ is a discrete family of open subsets of X so it follows from $ext(X) \leqslant 2^\kappa$ that $|\mathcal{P}| \leqslant 2^\kappa$ and hence $|Y| \leqslant 2^\kappa$. If Z is the set Y with the discrete topology and $g(x) = f(x)$ for any $x \in X$, then the map $g : X \to Z$ is continuous and, for the identity map $h : Z \to Y$, we have $h \circ g = f$. Since there exists an injection of Z into $\mathbb{I}^\kappa$, which is automatically continuous, we conclude that the maps g and h witness exponential κ-cofinality of X.

Example 42.

For any cardinal $\kappa > \mathfrak{c}$, consider the set $X = \kappa \cup \{p\}$ where $p \notin \kappa$. All points of κ are isolated in X and a set $U \subset X$ with $p \in U$ is open if and only if $\kappa \backslash U \leqslant \mathfrak{c}$. It is immediate that X is a $P_{\mathfrak{c}}$-space and $l(X) = ext(X) = \mathfrak{c}$ so X is exponentially ω-cofinal by Proposition 41. Therefore the Souslin number of an exponentially ω-cofinal space can be arbitrarily large. This result should be compared with Corollary 35.

We have already mentioned the equalities $iw(X) = d(C_p(X))$ and $d(X) = iw(C_p(X))$ for any space X, which show that the density and i-weight are bidual with respect to the functor C_p. Since exponential κ-domination and exponential κ-cofinality are their respective weakenings, it is natural to expect them to be bidual as well. We will see next that this is, indeed, the case.

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Theorem 43.

A space X is exponentially κ-cofinal if and only if $C_p(X)$ is a space with exponential κ-domination.

Proof.

Suppose that X is exponentially κ-cofinal and take a set $A \subset C_p(X)$ with $|A| \leqslant 2^\kappa$. If $u = \Delta A$ is the diagonal product of the family A, then $u : X \to \mathbb{R}^A$ and hence the space $Y = u(X)$ has weight not exceeding 2^κ. We can consider that $u : X \to Y$ and hence the dual map $u^* : C_p(Y) \to C_p(X)$ is an embedding. Let $Q = u^*(C_p(Y))$ and observe that $A \subset Q$.

Proof.

Suppose that X is exponentially κ-cofinal and take a set $A \subset C_p(X)$ with $|A| \leqslant 2^\kappa$. If $u = \Delta A$ is the diagonal product of the family A, then $u : X \to \mathbb{R}^A$ and hence the space $Y = u(X)$ has weight not exceeding 2^κ. We can consider that $u : X \to Y$ and hence the dual map $u^* : C_p(Y) \to C_p(X)$ is an embedding. Let $Q = u^*(C_p(Y))$ and observe that $A \subset Q$.

There exists a space Z together with continuous onto maps $v : X \to Z$ and $w : Z \to Y$ such that $iw(Z) \leqslant \kappa$ and $w \circ v = u$. The space $C_p(Z)$ has density not exceeding κ; since $E = v^*(C_p(Z))$ is homeomorphic to $C_p(Z)$, we can fix a set $B \subset E$ such that $|B| \leqslant \kappa$ and $E \subset \overline{B}$. Then $Q \subset E$ whence $A \subset Q \subset E \subset \overline{B}$ so the set B witnesses exponential κ-domination in $C_p(X)$.

To prove sufficiency, assume that $C_p(X)$ features exponential κ-domination and take a continuous onto map $u : X \to Y$ for some space Y of weight not exceeding 2^κ. The dual map $u^* : C_p(Y) \to C_p(X)$ is an embedding so the density of the set $Q = u^*(C_p(Y))$ is the same as the density of $C_p(Y)$ while $d(C_p(Y)) \leqslant nw(C_p(Y)) = nw(Y) \leqslant 2^\kappa$. This makes it possible to take a set $B \subset C_p(X)$ such that $|B| \leqslant \kappa$ and $Q \subset \overline{B}$. The reflection map $e_{\overline{B}} : X \to C_p(\overline{B})$ is continuous and the i-weight of space $Z = e_{\overline{B}}(X) \subset C_p(\overline{B})$ does not exceed $iw(C_p(\overline{B})) \leqslant |B| \leqslant \kappa$. The dual map $\varphi = e^*_{\overline{B}} : C_p(Z) \to C_p(X)$ is an embedding and $\overline{B} \subset \varphi(C_p(Z))$. Therefore $Q \subset \overline{B} \subset \varphi(C_p(Z))$ so there exist continuous onto maps $v : X \to Z$ and $w : Z \to Y$ such that $u = w \circ v$. Since $iw(Z) \leqslant \kappa$, the space Z witnesses that X is exponentially κ-cofinal.

Theorem 44.

Let κ be an infinite cardinal. A space X features exponential κ-domination if and only if $C_p(X)$ is exponentially κ-cofinal.

Theorem 44.

Let κ be an infinite cardinal. A space X features exponential κ-domination if and only if $C_p(X)$ is exponentially κ-cofinal.

Proof.

Assume that X is a space with exponential κ-domination and $u : C_p(X) \to Y$ is a continuous onto map for some space Y with $w(Y) \leqslant 2^\kappa$. There exists a set $A \subset X$ and a continuous onto map $\varphi : \pi_A(C_p(X)) \to Y$ such that $|A| \leqslant 2^\kappa$ and $\varphi \circ \pi_A = u$. By exponential κ-domination of X there exists a set $B \subset X$ such that $|B| \leqslant \kappa$ and $A \subset \overline{B}$. It is standard that the restriction map $\pi_A : C_p(X) \to \pi_A(C_p(X))$ factorizes through $\pi_{\overline{B}}(C_p(X))$ so there exists a continuous onto map $w : Z = \pi_{\overline{B}}(C_p(X)) \to Y$ such that $u = w \circ \pi_{\overline{B}}$. Since $iw(Z) \leqslant iw(C_p(\overline{B})) = d(\overline{B}) \leqslant |B| \leqslant \kappa$, we conclude that Z witnesses exponential κ-cofinality of $C_p(X)$.

Now assume that $C_p(X)$ is exponentially κ-cofinal and $A \subset X$ is a set with $|A| \leqslant 2^\kappa$. By Theorem 43 the space $C_pC_p(X)$ features exponential κ-domination so there is a set $E \subset C_pC_p(X)$ such that $|E| \leqslant \kappa$ and $A \subset \overline{E}$. Here we identify X with its canonical copy in $C_pC_p(X)$. Every continuous real-valued function on $C_p(X)$ depends on countably many coordinates so we can choose, for each $u \in E$, a countable set $B_u \subset X$ such that $u(f) = u(g)$ whenever $f, g \in C_p(X)$ and $f|B_u = g|B_u$.

The set $B = \bigcup\{B_u : u \in E\} \subset X$ has cardinality not exceeding κ; assume that $p \in A \backslash \overline{B}$. There exists a function $f \in C_p(X)$ such that $f(p) = 1$ and $f(\overline{B}) \subset \{0\}$; let $g(x) = 0$ for any $x \in X$. It follows from $p \in \overline{E}$ that there is a function $u \in E$ such that $u(f) > \frac{1}{2}$ and $u(g) < \frac{1}{2}$. However, $f|B = g|B$ and hence $f|B_u = g|B_u$ which implies that $u(f) = u(g)$. This contradiction shows that $A \subset \overline{B}$, i.e., the set B witnesses that X is a space with exponential κ-domination.

Corollary 45.

Given a cardinal $\kappa \geqslant \omega$, if X features exponential κ-domination, then $C_{p,2n}(X)$ is a space with exponential κ-domination and $C_{p,2n+1}(X)$ is exponentially κ-cofinal for all $n \in \omega$.

Corollary 45.

Given a cardinal $\kappa \geqslant \omega$, if X features exponential κ-domination, then $C_{p,2n}(X)$ is a space with exponential κ-domination and $C_{p,2n+1}(X)$ is exponentially κ-cofinal for all $n \in \omega$.

Proof.

It follows from Theorem 44 that $C_p(X)$ must be exponentially κ-cofinal and hence Theorem 43 can be applied to see that $C_pC_p(X)$ is a space with exponential κ-domination. Proceeding by induction assume that $C_{p,2n}(X)$ is a space with exponential κ-domination. Then $C_{p,2n+1}(X) = C_p(C_{p,2n}(X))$ is an exponentially κ-cofinal space by Theorem 44 so Theorem 43 shows that $C_{p,2n+2}(X) = C_p(C_{p,2n+1}(X))$ features exponential κ-domination.

Corollary 46.

If X is an exponentially κ-cofinal space, then $C_{p,2n+1}(X)$ is a space with exponential κ-domination and $C_{p,2n}(X)$ is exponentially κ-cofinal for all $n \in \omega$.

Corollary 46.

If X is an exponentially κ-cofinal space, then $C_{p,2n+1}(X)$ is a space with exponential κ-domination and $C_{p,2n}(X)$ is exponentially κ-cofinal for all $n \in \omega$.

Proof.

By Theorem 43, the space $Y = C_p(X)$ has exponential κ-domination; apply Corollary 45 to convince ourselves that $C_{p,2n+1}(X) = C_{p,2n}(Y)$ is a space with exponential κ-domination and $C_{p,2n}(X) = C_{p,2n-1}(Y)$ is an exponentially κ-cofinal space.

Theorem 47.

Suppose that X is an exponentially κ-cofinal space such that $l(X) \leqslant \kappa$ and $t(X) \leqslant \kappa$. Then $iw(X) \leqslant \kappa$ and hence $|X| \leqslant 2^\kappa$.

Theorem 47.

Suppose that X is an exponentially κ-cofinal space such that $l(X) \leqslant \kappa$ and $t(X) \leqslant \kappa$. Then $iw(X) \leqslant \kappa$ and hence $|X| \leqslant 2^\kappa$.

Proof.

We will prove first that $\psi(X) \leqslant \kappa$. Striving for contradiction, assume that $p \in X$ and $\psi(p, X) > \kappa$. Take any $x_0 \in X \backslash \{p\}$ and let $G_0 = X$. Proceeding by induction, assume that $\beta < \kappa^+$ and we have a set $\{x_\alpha : \alpha < \beta\} \subset X \backslash \{p\}$ and a family $\{G_\alpha : \alpha < \beta\}$ of closed G_κ-subsets of X with the following properties:

(1) $\{p, x_\alpha\} \subset G_\alpha$ for all $\alpha < \beta$;

(2) $G_\alpha \subset G_\gamma$ whenever $\gamma < \alpha < \beta$;

(3) $\overline{\{x_\gamma : \gamma < \alpha\}} \cap G_\alpha \subset \{p\}$ for every $\alpha < \beta$.

The set $B_\beta = \{p\} \cup \overline{\{x_\alpha : \alpha < \beta\}}$ has cardinality $\leqslant 2^\kappa$ by Corollary 37 which implies, by Theorem 36, that $\psi(p, B_\beta) \leqslant \kappa$ and hence we can choose a closed G_κ-set Q in the space X such that $Q \cap B_\beta = \{p\}$. Let $G_\beta = \bigcap\{G_\alpha : \alpha < \beta\} \cap Q$; since $\psi(p, X) > \kappa$, we can pick a point $x_\beta \in G_\beta \backslash \{p\}$ completing our inductive construction. Observe that it follows from the properties (1) and (3) that the set $A = \{x_\alpha : \alpha < \kappa^+\}$ is faithfully indexed and hence $|A| = \kappa^+$.

Let $F_\alpha = \overline{\{x_\beta : \alpha \leqslant \beta < \kappa^+\}} \subset G_\alpha$ for any $\alpha < \kappa^+$; it follows from $l(X) \leqslant \kappa$ that $\emptyset \neq F = \bigcap\{F_\alpha : \alpha < \kappa^+\}$. If $x \neq p$ and $x \in F$, then $x \in \overline{A}$; since $t(X) \leqslant \kappa$, there exists $\beta < \kappa^+$ such that $x \in \overline{\{x_\alpha : \alpha < \beta\}}$. Since also $x \in F_\beta \subset G_\beta$, we obtained a contradiction with the property (3). Thus, $F = \{p\}$ and it is standard to deduce from $l(X) \leqslant \kappa$ that

(4) the family $\{F_\alpha : \alpha < \kappa^+\}$ is a network at p, i.e., for any set $U \in \tau(p, X)$ there exists $\alpha < \kappa^+$ such that $F_\alpha \subset U$ and hence $\{x_\beta : \alpha < \beta\} \subset U$.

It is an immediate consequence of the property (4) that A is concentrated around the point p and hence the set $D = \{(p, x_\alpha) : \alpha < \kappa^+\} \subset (X \times X) \backslash \Delta_X$ is concentrated around the diagonal Δ_X. Since $|D| = \kappa^+$, the diagonal of X is not κ^+-small; this contradiction with Proposition 40 shows that $\psi(X) \leqslant \kappa$. Arhangel'skii's inequality $|X| \leqslant 2^{l(X)\cdot\psi(X)\cdot t(X)} \leqslant 2^\kappa$ shows that $|X| \leqslant 2^\kappa$ and therefore $iw(X) \leqslant \kappa$ by Theorem 36.

Corollary 48.

If X is a Lindelöf exponentially ω-cofinal space and $t(X) \leqslant \omega$, then $iw(X) \leqslant \omega$.

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If X is a Lindelöf exponentially ω-cofinal space and $t(X) \leqslant \omega$, then $iw(X) \leqslant \omega$.

Corollary 49.

Suppose that X is a space for which $C_p(X)$ features exponential κ-domination while $l(C_p(X)) \leqslant \kappa$ and $t(C_p(X)) \leqslant \kappa$. Then $d(C_p(X)) \leqslant \kappa$.

Corollary 48.

If X is a Lindelöf exponentially ω-cofinal space and $t(X) \leqslant \omega$, then $iw(X) \leqslant \omega$.

Corollary 49.

Suppose that X is a space for which $C_p(X)$ features exponential κ-domination while $l(C_p(X)) \leqslant \kappa$ and $t(C_p(X)) \leqslant \kappa$. Then $d(C_p(X)) \leqslant \kappa$.

Proof.

The space X is exponentially κ-cofinal by Theorem 43. Next, observe that $t(X) \leqslant l(C_p(X)) \leqslant \kappa$ and $l(X) \leqslant t(C_p(X)) \leqslant \kappa$ which shows that Theorem 47 can be applied to see that $iw(X) \leqslant \kappa$ whence $d(C_p(X)) = iw(X) \leqslant \kappa$.

Proposition 50.

Assume that X is a κ-stable exponentially κ-cofinal space. Then $nw(X) \leqslant \kappa$.

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Assume that X is a κ-stable exponentially κ-cofinal space. Then $nw(X) \leqslant \kappa$.

Proof.

If $d(C_p(X)) > \kappa$, then there exists a left-separated set $L \subset C_p(X)$ with $|L| = \kappa^+$. Next, observe that $C_p(X)$ is a space with exponential κ-domination by Theorem 43 and hence there exists a set $A \subset C_p(X)$ such that $|A| \leqslant \kappa$ and $L \subset \overline{A}$. The space $C_p(X)$ is κ-monolithic so $nw(\overline{A}) \leqslant \kappa$. Therefore $\kappa^+ \leqslant hd(L) \leqslant nw(L) \leqslant nw(\overline{A}) \leqslant \kappa$; this contradiction shows that $d(C_p(X)) \leqslant \kappa$ and therefore $iw(X) = d(C_p(X)) \leqslant \kappa$. Finally, apply κ-stability of X to conclude that $nw(X) \leqslant \kappa$.

Corollary 51.

Any exponentially ω-cofinal Lindelöf Σ-space has a countable network.

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Corollary 52.

Any exponentially ω-cofinal pseudocompact space is compact and metrizable.

Example 53.

For any cardinal $\kappa > \mathfrak{c}$, there exists an exponentially ω-cofinal space Y such that $l(Y) \geqslant \kappa$. To see it, consider the exponentially ω-cofinal space $X = \kappa \cup \{p\}$ from Example 42. All points of κ are isolated in X and a set $U \subset X$ with $p \in U$ is open if and only if $\kappa \backslash U \leqslant \mathfrak{c}$. It is standard to see that $C_p(X)$ is homeomorphic to the $\Sigma_{\mathfrak{c}}$-product $S = \{x \in \mathbb{R}^\kappa : |x^{-1}(\mathbb{R}\backslash\{0\})| \leqslant \mathfrak{c}\}$ in the space $\mathbb{R}^\kappa$ and therefore S is a space with exponential ω-domination by Theorem 43. If we let $u(\alpha) = 1$ for any $\alpha < \kappa$, then we obtain a point $u \in \mathbb{R}^\kappa$ such that $u \notin \overline{A}$ for any $A \subset S$ with $|A| < \kappa$. This shows that the space $Z = S \cup \{u\}$ has tightness equal to κ. The union of countably many spaces with exponential ω-domination is easily seen to have exponential ω-domination so Z features exponential ω-domination. Observe that $l(C_p(Z)) \geqslant t(Z) = \kappa$ and therefore $l(C_p(Z)) \geqslant \kappa$. Finally, observe that $C_p(Z)$ is exponentially ω-cofinal by Theorem 44 and hence the space $Y = C_p(Z)$ is as promised.

5. Question.

Suppose that X is a Lindelöf exponentially ω-cofinal space. Is it true that $iw(X) \leqslant \omega$?

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Suppose that X is a Lindelöf exponentially ω-cofinal space. Is it true that $iw(X) \leqslant \omega$?

6. Question.

Suppose that X is a Lindelöf exponentially ω-cofinal space. Is it true that $\psi(X) \leqslant \omega$?

Open questions

5. Question.

Suppose that X is a Lindelöf exponentially ω-cofinal space. Is it true that $iw(X) \leqslant \omega$?

6. Question.

Suppose that X is a Lindelöf exponentially ω-cofinal space. Is it true that $\psi(X) \leqslant \omega$?

7. Question.

Suppose that X is a Lindelöf exponentially ω-cofinal space. Is it true that $|X| \leqslant 2^{\mathfrak{c}}$?

8. Question.

Suppose that X is an exponentially ω-cofinal space of countable character. Is it true that $iw(X) \leqslant \omega$?

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Suppose that X is an exponentially ω-cofinal space of countable character. Is it true that $iw(X) \leqslant \omega$?

9. Question.

Suppose that X is an exponentially ω-cofinal Fréchet–Urysohn space. Is it true that $iw(X) \leqslant \omega$?

8. Question.

Suppose that X is an exponentially ω-cofinal space of countable character. Is it true that $iw(X) \leqslant \omega$?

9. Question.

Suppose that X is an exponentially ω-cofinal Fréchet–Urysohn space. Is it true that $iw(X) \leqslant \omega$?

10. Question.

Let X be an exponentially ω-cofinal space with $\psi(X) \leqslant \omega$. Is it true that $iw(X) \leqslant \omega$?

11. Question.

Let X be an exponentially ω-cofinal space with a G_δ-diagonal. Is it true that $iw(X) \leqslant \omega$?

Open questions

11. Question.

Let X be an exponentially ω-cofinal space with a G_δ-diagonal. Is it true that $iw(X) \leqslant \omega$?

12. Question.

Suppose that X is an exponentially ω-cofinal space. Is it true that $ext(X) \leqslant \mathfrak{c}$?

11. Question.

Let X be an exponentially ω-cofinal space with a G_δ-diagonal. Is it true that $iw(X) \leqslant \omega$?

12. Question.

Suppose that X is an exponentially ω-cofinal space. Is it true that $ext(X) \leqslant \mathfrak{c}$?

13. Question.

Suppose that $C_p(X)$ is a Fréchet–Urysohn space that features exponential ω-domination. Must $C_p(X)$ be separable?

The results of this talk were published in the papers

G. Gruenhage, V.V. Tkachuk, R.G. Wilson, *Domination by small sets versus density,* Topology Appl., **282**(2020), 107306.

V.V. Tkachuk, *Exponential domination in function spaces,* CMUC, 61:3(2020), 397-408.

Thanks for your attention!!!