MAD families and strategically bounding forcings

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The *cardinal invariants of the continuum* are uncountable cardinals whose size is at most the cardinality of the real numbers. We are mostly interested in cardinals with a nice topological or combinatorial definition.

1. By ω we denote the set (cardinal) of the natural numbers.
2. By $\mathfrak{c}$ we denote the cardinality of the real numbers.

1. The cardinal invariants of the continuum are cardinals $\mathfrak{j}$ such that:

$$\omega < \mathfrak{j} \leq \mathfrak{c}$$

2. The *Continuum Hypothesis* (*CH*) is the following statement:

$\mathfrak{c}$ is the first uncountable cardinal

3. All cardinal invariants are $\mathfrak{c}$ under *CH.*
4. *Martin's Axiom (MA)* implies that most cardinal invariants are $\mathfrak{c}$.

The point is that the value of $\mathfrak{c}$ does not determine many of the combinatorial and topological properties of the “reals” $(\wp(\omega), 2^\omega, \omega^\omega, \mathbb{R}...)$. Let’s look at two models where $\mathfrak{c} = \omega_2$.

The Sacks model	A model of PFA
There is a non-meager set of size ω_1	Every set of size ω_1 is meager
There is a non-null set of size ω_1	Every set of size ω_1 has measure zero
ω^ω can be covered with ω_1-many meager sets	Union of ω_1-many meager sets is meager
$\mathbb{R}$ can be covered with ω_1-many null sets	Union of ω_1-many null sets has measure zero

In both models we have that $\mathfrak{c} = \omega_2$, however, the structure and properties of the reals are very different in those models. The value of the cardinal invariants in a model provide us a lot of information regarding the reals in such model.

Many of the cardinal invariants can be seen as the first moment where a "diagonalization argument fails". With this knowledge, we can carry some of the previous known constructions using CH to a different model.

Let $f, g \in \omega^\omega$, define $f \leq^* g$ if and only if $f(n) \leq g(n)$ holds for all $n \in \omega$ except finitely many. We say a family $\mathcal{B} \subseteq \omega^\omega$ is *unbounded* if $\mathcal{B}$ is unbounded with respect to $\leq^*$. We say that $\mathcal{D} \subseteq \omega^\omega$ is *dominating* if for every $f \in \omega^\omega$, there is $g \in \mathcal{D}$ such that $f \leq^* g$.

Definition

The *bounding number* $\mathfrak{b}$ is the size of the smallest unbounded family.

Definition

The *dominating number* $\mathfrak{d}$ is the size of the smallest of a dominating family.

Clearly, we have that $\mathfrak{b} \leq \mathfrak{d}$.

Lemma

$\mathfrak{b}$ *is uncountable.*

Proof.

We need to show that every countable subset of ω^ω is bounded. Let $\mathcal{B} = \{f_n \mid n \in \omega\}$, define $g \in \omega^\omega$ given by $g(n) = f_0(n) + ... + f_n(n)$. It is easy to see that g bounds $\mathcal{B}$. □

Obviously, the whole ω^ω is unbounded, so we get:

Corollary

$\omega < \mathfrak{b} \leq \mathfrak{c}$.

Definition

An infinite family $\mathcal{A} \subseteq [\omega]^{\omega}$ is *almost disjoint (AD)* if the intersection of any two different elements of $\mathcal{A}$ is finite. A MAD *family* is a maximal almost disjoint family.

Note that MAD families exists under the Axiom of Choice (in fact, every AD family can be extended to a MAD family). There are models of ZF where there is no MADness.

Definition

The *almost disjointness number* $\mathfrak{a}$ is the smallest size of a MAD family.

Lemma

$\mathfrak{a}$ *is an uncountable cardinal.*

We need to prove that there are no countable MAD families. Let $\mathcal{A} = \{A_n \mid n \in \omega\}$ be an AD family. For every $n \in \omega$, we choose $b_n \in A_n \setminus \bigcup_{i<n} A_i$. Let $B = \{b_n \mid n \in \omega\}$, it follows that B is almost disjoint with every element of $\mathcal{A}$.

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- We already know that $\mathfrak{b} \leq \mathfrak{d}$.
- It is not hard to prove that $\mathfrak{b} \leq \mathfrak{a}$.

In fact, we can think of $\mathfrak{a}$ as the "AD-version of $\mathfrak{b}$".

Given $n \in \omega$, define $C_n = \{n\} \times \omega$.

$\mathfrak{b}$ is the smallest size of a family $\mathcal{B} \subseteq \omega \times \omega$ with the following properties:

1. Every element of $\mathcal{B}$ is almost disjoint with every C_n.
2. For every $X \in [\omega]^\omega$ and $f : X \longrightarrow \omega$, there is $B \in \mathcal{B}$ such that $B \cap f$ is infinite (we view f as a subset of $\omega \times \omega$).

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What about $\mathfrak{a}$ and $\mathfrak{d}$?

Theorem (Kunen?)

There is a model in ZFC in which $\mathfrak{a} < \mathfrak{d}$. In fact, such inequality holds in the Cohen model.

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Yes! But it is MUCH harder.

In order to build a model of $\mathfrak{d} < \mathfrak{a}$, Shelah developed the techniques of forcing along a template.

Theorem (Shelah)

Assume GCH. Let κ and μ be regular cardinals with $\omega_1 < \kappa < \mu$. There is a ccc extension in which $\mathfrak{b} = \mathfrak{d} = \kappa$ and $\mathfrak{a} = \mathfrak{c} = \mu$.

In particular, we get the following:

Theorem (Shelah)

There is a model of ZFC in which $\omega_2 = \mathfrak{d} < \mathfrak{a} = \omega_3$.

The theorem of Shelah has an interesting feature, $\mathfrak{d}$ can be any regular cardinal except ω_1. The natural question is the following:

Problem (Roitman)

Does $\mathfrak{d} = \omega_1$ *imply* $\mathfrak{a} = \omega_1$*?*

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It would be weird if $\mathfrak{d} = \omega_1$ implied $\mathfrak{a} = \omega_1$ (given that this is not true for any other regular cardinal)... but ω_1 is weird cardinal, it simply behaves differently than the other regular cardinals. Every time I become more convinced that a technique of Todorcevic could be using to build a small MAD family from a small dominating family.

Are there known examples of this phenomenon?

Are there two cardinal invariants $\mathfrak{j}_1$ and $\mathfrak{j}_2$ such that $\mathfrak{j}_2 < \mathfrak{j}_1$ is consistent, yet $\mathfrak{j}_2 = \omega_1$ imply $\mathfrak{j}_1 = \omega_1$?

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Yes! We will see an example.

$\mathfrak{a}_s$ is the smallest size of a family $\mathcal{A} \subseteq \omega \times \omega$ with the following properties:

1. Every element of $\mathcal{A}$ is an infinite partial function from ω to ω.
2. For every $X \in [\omega]^{\omega}$ and $f : X \longrightarrow \omega$, there is $g \in \mathcal{A}$ such that $g \cap f$ is infinite.
3. $\mathcal{A}$ is an AD family.

By $\text{non}(\mathcal{M})$ we denote the smallest size of non-meager subset of ω^{ω}.

1. $\max\{\text{non}(\mathcal{M}), \mathfrak{a}\} \leq \mathfrak{a}_s$.
2. (Brendle) It is consistent that $\omega_2 = \max\{\text{non}(\mathcal{M}), \mathfrak{a}\} < \mathfrak{a}_s$.
3. (G., Hrušák, Téllez) $\max\{\text{non}(\mathcal{M}), \mathfrak{a}\} = \omega_1$ implies $\mathfrak{a}_s = \omega_1$.

In this way, $\max\{\text{non}(\mathcal{M}), \mathfrak{a}\}$ and $\mathfrak{a}_s$ may be different, but not if $\max\{\text{non}(\mathcal{M}), \mathfrak{a}\}$ is ω_1.

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Problem

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1. A forcing is ω^ω-bounding it it does not add unbounded reals (i.e. $\omega^\omega \cap V$ is still a dominating family after forcing with $\mathbb{P}$).
2. A forcing $\mathbb{P}$ destroys a MAD family $\mathcal{A}$ if $\mathcal{A}$ is no longer maximal after forcing with $\mathbb{P}$.
3. If $\mathbb{P}$ does not destroy $\mathcal{A}$, we say that $\mathcal{A}$ is $\mathbb{P}$-indestructible.

The problem of Roitman is probably equivalent to the following:

Problem

Assume CH. Let $\mathcal{A}$ be a MAD family. Is there a proper ω^ω-bounding forcing that destroys $\mathcal{A}$?

If the answer to the problem is "yes", we can perform a forcing iteration yielding a model of $\omega_1 = \mathfrak{d} < \mathfrak{a}$.

Theorem (Shelah)

The countable support iteration of proper ω^ω-bounding forcings is ω^ω-bounding.

Problem

Assume CH. Is there a MAD family that is indestructible under any proper ω^ω-bounding forcing?

There has been many advances in this problem (suggesting a positive answer?).

Theorem (Garcia-Ferreira, Hrušák)

Assume $V \models CH$. Let $\mathbb{P}$ be proper ω^ω-bounding forcing of size ω_1. There is a $\mathbb{P}$-indestructible MAD family.

In this way, no proper ω^ω-bounding forcing of size ω_1 can take care of all MAD families.

Theorem (Džamonja, Hrušák, Moore)

Let $\langle \mathbb{P}_\alpha \rangle_{\alpha<\omega_2}$ be a sequence of Borel partial orders such that each $\mathbb{P}_\alpha$ is of the form $\wp(2)^+ \times \mathbb{Q}_\alpha$ for some $\mathbb{Q}_\alpha$. Let $\mathbb{P}$ be the countable support iteration of the sequence. If $\mathbb{P}$ is proper and ω^ω-bounding, then "$\mathfrak{a} = \omega_1$" holds after forcing with $\mathbb{P}$.

In some sense, the theorem above says that in order to get a model of $\mathfrak{b} < \mathfrak{a}$, we need to use non-definable forcings.

Theorem (Laflamme)

If a MAD family can be extended to an F_σ-ideal, then it can be destroyed by a proper ω^ω-bounding forcing. However, under CH there are MAD families that can not be extended to an F_σ-ideal.

Definition

Let $\mathcal{A}$ be an AD family. By $\mathcal{I}(\mathcal{A})$ we denote the ideal generated by $\mathcal{A}$ (and all finite subsets of ω).

Definition

Let $\mathcal{A}$ be a MAD family. We say that $\mathcal{A}$ is *Shelah-Steprāns* if for every $X \subseteq [\omega]^{<\omega} \setminus \{\emptyset\}$, there is $B \in \mathcal{I}(\mathcal{A})$ such that one of the following conditions hold:

1. $B \cap s \neq \emptyset$ for every $s \in X$, or
2. B contains infinitely many elements of X.

Shelah-Steprāns MAD families have very strong combinatorial properties.

Theorem (Raghavan)

It is consistent that there are no Shelah-Steprāns MAD families.

On the other hand,

Theorem (Brendle,G., Hrušák, Raghavan)

Both $\mathfrak{p} = \mathfrak{c}$ and $\Diamond(\mathfrak{b})$ imply that there are Shelah-Steprāns MAD families.

We discovered that Shelah-Steprāns MAD families are very indestructible.
It might be the case that Shelah-Steprāns MAD families are indestructible by every proper ω^ω-bounding forcings.

Theorem (Brendle, G., Hrušák, Raghavan)

(LC) Let $\mathcal{A}$ be a Shelah-Steprāns MAD family and $\mathcal{J}$ a "definable" σ-ideal in ω^ω such that $\mathbb{P}_{\mathcal{J}} = \text{Borel}\,(\omega^\omega)\,/\,\mathcal{J}$ is proper and has the continuos reading of names. If $\mathbb{P}_{\mathcal{J}}$ destroys $\mathcal{A}$, then it adds a dominating real.

Let $\mathbb{P}$ be a partial order and $p \in \mathbb{P}$. We define the *bounding game* $\mathcal{BG}(\mathbb{P}, p)$ as follows:

I	D_0		D_1		...
II		B_0		B_1	...

Where each $D_n \subseteq \mathbb{P}$ is open dense below p and $B_n \in [D_n]^{<\omega}$. Player II will *win the game* if there is $q \leq p$ such that B_n is predense below q for every $n \in \omega$ (i.e. if every $r \leq q$ is compatible with an element of B_n).

Theorem (Zapletal)

Let $\mathbb{P}$ be a proper forcing. The following are equivalent:

1. $\mathbb{P}$ *is* ω^{ω}*-bounding.*
2. *For every* $p \in \mathbb{P}$*, the player* I *does not have a winning strategy on* $\mathcal{BG}(\mathbb{P}, p)$.

This result can be used as motivation for the following definition:

Definition

Let $\mathbb{P}$ be a partial order. $\mathbb{P}$ is *strategically bounding* if for every $p \in \mathbb{P}$, the player II has a winning strategy on $\mathcal{BG}(\mathbb{P}, p)$.

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Examples of strategically bounding forcings are the Sacks, Silver and random forcings. In fact, the usual proofs that these forcings are ω^ω-bounding actually show that they are strategically bounding.

Strategically bounding forcings have been studied in the past. In particular, the ccc case has received a lot of attention because of its relation with Maharam's and von Neumann's problems. The following is a very important result of Fremlin:

Theorem (Fremlin)

Let $\mathbb{B}$ be a ccc complete Boolean algebra. the following are equivalent:

1. $\mathbb{B}$ *is strategically bounding.*
2. *There is a continuous submeasure on the algebra* $\mathbb{B}$.

Some strategically bounding forcings are of the following form:

Definition

Let $\mathbb{P}$ be a partial order. We say that $\mathbb{P}$ is *axiom A for* $\mathfrak{d}$ (or *has an axiom A structure for* $\mathfrak{d}$) if there is a sequence of partial orders $\langle \leq_n \rangle_{n \in \omega}$ with the following properties:

1. If $p \leq_0 q$ then $p \leq q$.
2. If $p \leq_{n+1} q$ then $p \leq_n q$ for every $n \in \omega$.
3. (*Fusion property*) If $\langle p_n \rangle_{n \in \omega}$ is a sequence such that $p_{n+1} \leq_n p_n$ for every $n \in \omega$, then there is $q \in \mathbb{P}$ such that $q \leq_n p_n$ for every $n \in \omega$.
4. (*Bounding Freezing property*) For every $p \in \mathbb{P}$, $A \subseteq \mathbb{P}$ a maximal antichain and $n \in \omega$, there is $q \leq_n p$ such that $\{r \in A \mid r \text{ and } q \text{ are compatible}\}$ is finite.

Theorem (G., Hrušák)

The countable support iteration of proper strategically bounding forcings is strategically bounding.

Theorem (G., Hrušák)

If $\mathcal{A}$ is a Shelah-Steprāns MAD family and $\mathbb{P}$ a strategically bounding forcing, then $\mathcal{A}$ is $\mathbb{P}$-indestructible.

Thank you very much!