Extensions of Inner Models of ZFC

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Results of Vopěnka's seminary in Prague, mainly of the 1960's and 1970's. The results were often published in terminology of Semiset Theory. That was reason why many set theorists did not read those results in spite of the fact, that the translation of them to the set theory language is rather trivial.
2017 L.B. in Comment. Math. Univ. Carolinae, in which I needed some results about support. The reviewer wrote:
"The Czech's school's work in Set Theory in the 1960's and 1970's is not readily accessible in the west. Personally, I found the paper both useful and interesting."

Introduction

L.B., *Extensions of Inner Models of* ZFC, In: Research Trends in Contemporary Logic, (D. Gabbay, M. Fitting, M. Pourmahdian, A.Rezus, and A.S. Daghighi, eds.), College Publications London, to appear.

The results were presented in the set theory language.

Notations and Terminology

The terminology and notations are those of [J] T. Jech, Set Theory, Springer 2003, with the exception that by **inner model** M we understand transitive class such that $Ord \subseteq M$ and $(M, \in)$ is a model of ZFC.

If M, N are inner models and $M \subseteq N$, we say that N is **an extension of** M.

Often we shall work with an inner model M and the extension $M \subseteq V$. Indeed, if N is an inner model then $V^N = N$

If M is an inner model, $x \subseteq M$ is a set, then $M[x]$ is the least inner model such that $M \subseteq M[x]$ and $x \in M[x]$, compare [J], Exercise 13.34.

Theorem 1 (B. Balcar and P. Vopěnka 1967)

$M \subseteq N_1$, $M \subseteq N_2$ are extensions. If $\mathcal{P}(Ord) \cap N_1 = \mathcal{P}(Ord) \cap N_2$ then $N_1 = N_2$.

Corollary 2

If M is an inner model then the universe V is uniquely determined by $\mathcal{P}(M)$.

That was the idea of **The Theory of Semisets:** element $x \in M$ is **a set**, subset $\rho \subseteq M$ is **a semiset.**

Notations and Terminology

M is an inner model, $\tau, \sigma \subseteq M$. **The set τ is dependent on the set σ over M**, written $\mathrm{Dep}_M(\tau, \sigma)$, if there exists a binary relation $r \in M$ such that

$$\tau = r''\sigma = \{y \in \mathrm{rng}(r) : (\exists x \in \sigma)\,(x, y) \in r\}.$$

For any sets $\tau, \sigma, \rho \subseteq M$ we have

$$(\mathrm{Dep}_M(\tau, \sigma) \wedge \mathrm{Dep}_M(\sigma, \rho)) \to \mathrm{Dep}_M(\tau, \rho).$$

If $\tau = r_1''\sigma$ and $\sigma = r_2''\rho$ then $\tau = (r_1 \circ r_2)''\rho$.

Support

$$(\mathrm{Dep}_M(\tau,\sigma) \wedge \mathrm{Dep}_M(\rho,\sigma)) \to \mathrm{Dep}_M(G_i(\tau,\rho),\sigma) \quad (1)$$

for $i = 6 - 10$, where G_i are Gödel operations.
Note that (1) has no sense for $G_1(\tau,\rho) = \{\tau,\rho\}$.
A set $\sigma \subseteq M$ is **a support over** M if for any sets $\tau, \rho \subseteq M$

$$(\mathrm{Dep}_M(\tau,\sigma) \wedge \mathrm{Dep}_M(\rho,\sigma)) \to \mathrm{Dep}_M(\tau \setminus \rho,\sigma).$$

If σ is a support then (1) holds true for $i = 2 - 10$.
σ is **a total support of** N **over** M if every $\tau \in N$, $\tau \subseteq M$ is $\mathrm{Dep}_M(\tau,\sigma)$.
A total support is a support.

Generic Extension

Let $M \subseteq N$ be a generic extension. I.e.:
$(P, \leq)$ is a forcing in M, $e : P \longrightarrow B(P)$ is the embedding of P into cBa $B(P)$, see [J], p. 206, $G \subseteq B(P)$ a filter generic over M and $N = M[G]$.
For $\tau \subseteq A \in M$, $\tau \in N$, there exists a function $f : A \longrightarrow B(P)$, $f \in M$, the name of τ, such that

$$\tau = \{x \in A : f(x) \in G\} = f_{-1}(G).$$

If $\sigma = P \cap G$, then $\tau = r``\sigma$, where

$$r = \{(p, x) : p \in P \wedge x \in A \wedge e(p) \leq f(\check{x})\}.$$

Both σ and G are total supports of N over M.

Support

Lemma 3

If $\sigma \subseteq P \in M$ is a support over M, then there exists a relation $r \in M$ such that

$$\mathcal{P}(P \setminus \sigma) \cap M = r''\sigma.$$

Proof. $\mathcal{P}(P) \cap M = r_1''\sigma$ where $p \in \sigma$ is fix and
$r_1 = \{(p, u) : u \in \mathcal{P}(P) \cap M\}$.
$\{u \subseteq P : u \in M \wedge \sigma \cap u \neq \emptyset\} = r_2''\sigma$ where
$r_2 = \{(p, u) : p \in u \subseteq P \wedge u \in M\}$.
Since σ is a support, there exists a relation $r \in M$

$$r''\sigma = r_1''\sigma \setminus r_2''\sigma = \mathcal{P}(P \setminus \sigma) \cap M. \qquad \square$$

Generic Extension

If $M \subseteq M[\sigma]$ is a generic extension then the equation of Lemma 3 holds true with

$$r = \{(p,u) \in P \times \mathcal{P}(P) \cap M : (\forall q \in u)\, p \wedge q = 0\}.$$

In Boolean case

$$\mathcal{P}(B \setminus G) \cap M = f_{-1}(G),$$

where $f \in M$ and

$$f(u) = -\sum u$$

for $u \subseteq B(P)$, $u \in M$.

Normal Form of a Support

Theorem 4 (P. Vopěnka 1972)

If $\sigma \subseteq M$ is a support then $M \subseteq M[\sigma]$ is a generic extension.

Beautiful Balcar's Theorem about
Normal form of a support.

Theorem 5 (B. Balcar 1973)

If $\sigma \subseteq P \in M$ is a support then there exist an equivalence relation $\sim \in M$ on P and a partial order $\leq \in M$ such that $\sigma / \sim$ is a filter on $(P/\sim, \leq)$ generic over M.

Support

Following Balcar's proof in 2018, I succeeded to prove an addition to Balcar's Theorem 5.

Theorem 6 (L.B. 2018)

If there exists a function $f \in M$ such that $\mathcal{P}(P \setminus \sigma) \cap M = f_{-1}(\sigma)$, then there exist an equivalence relation $\sim \in M$ on P and a partial order $\leq \in M$ such that $(P/\sim, \leq)$ is cBa in M, $\sigma/\sim$ is a filter on $(P/\sim, \leq)$ generic over M. Moreover, for every $u \subseteq P$, $u \in M$ we have

$$[f(u)] = -\sum\{[x] : x \in u\}.$$

Support

Theorem 7 (The Product Lemma)

If $M_2 = M_1[\sigma_1]$ and $M_3 = M_2[\sigma_2]$, where $\sigma_1 \subseteq M_1$ is a support over M_1 and $\sigma_2 \subseteq M_1$ is a support over M_2, then $\sigma_1 \times \sigma_2$ is a support over M_1 and $M_3 = M_1[\sigma_1 \times \sigma_2]$.

Proof. Let $\tau \subseteq M_1$, $\tau \in M_3$. Then $\tau = r_2 " \sigma_2$ and $r_2 = r_1 " \sigma_1$ for some $r_1 \in M_1$ and $r_2 \in M_2$. Thus

$$x \in \tau \equiv (\exists y \in \sigma_2)\,(y, x) \in r_2 \equiv$$
$$(\exists y \in \sigma_2)(\exists z \in \sigma_1)(z, (y, x)) \in r_1.$$

Let $r = \{((z, y), x) : (z, (y, x)) \in r_1\}$. Then

$$\tau = r" \sigma_1 \times \sigma_2.$$

Proof of Theorem 5. By Lemma 3 there is $r \in M$ such that $\mathcal{P}(P \setminus \sigma) \cap M = r''\sigma$. Define the "incompatibility" relation

$$R = \{(p,q) \in P \times P : (\exists u \in M)(((p,u) \in r \wedge q \in u) \vee ((q,u) \in r \wedge p \in u)) \wedge p \neq q\}.$$

Then $R \in M$ and we have:

(i) R is a symmetric antireflexive relation on P.

(ii) $R''\{p\} \subseteq P \setminus \sigma$ for any $p \in \sigma$.

(iii) For any $u \subseteq P \setminus \sigma$, $u \in M$, there exists a $p \in \sigma$ such that $u \subseteq R''\{p\}$.

Proof of Theorem 5

(i) is by definition.

(ii) Let $q \in R''\{p\}$, $p \in \sigma$. Then there exists $u \subseteq P$, $u \in M$, such that
either: $q \in u$ and $(p, u) \in r$. Since $p \in \sigma$ we obtain $u \subseteq P \setminus \sigma$. Hence $q \in P \setminus \sigma$.
or: $p \in u$ and $(q, u) \in r$. Since $u \not\subseteq P \setminus \sigma$ we obtain $q \notin \sigma$.

(iii) Let $u \subseteq P \setminus \sigma$, $u \in M$. By the Lemma there exists a $p \in \sigma$ such that $(p, u) \in r$. Then $(p, q) \in R$ for every $q \in u$. So $u \subseteq R''\{p\}$.

Proof of Theorem 5

Set

$$p \sim q \equiv R``\{q\} = R``\{p\},\ p \leq q \equiv R``\{q\} \subseteq R``\{p\}.$$

Claim: σ is a filter on $(P, \leq)$ generic over M:
If $p, q \in \sigma$ then $R``\{p\} \cup R``\{q\} \subseteq P \setminus \sigma$. By (iii) there exists a $t \in \sigma$ such that $R``\{p\} \cup R``\{q\} \subseteq R``\{t\}$. Then $t \leq p$ and $t \leq q$.
We show $(p \leq q \wedge p \in \sigma) \to q \in \sigma$. Assume $q \notin \sigma$.
By (iii) there exists a $t \in \sigma$ such that $R``\{p\} \cup \{q\} \subseteq R``\{t\}$. Then $(t, q) \in R$, hence $(q, t) \in R$ as well. Since $R``\{q\} \subseteq R``\{p\} \subseteq P \setminus \sigma$, we obtain $t \in P \setminus \sigma$ – a contradiction.

Proof of Theorem 5

Let $u \subseteq P$, $u \in M$. Assume $u \cap \sigma = \emptyset$. We show that u is not dense in $(P/\sim, \leq)$..
By (iii) there exists an $p \in \sigma$ such that $u \subseteq R``\{p\}$.
Claim p is incompatible with every element of u.
Assume not. Then there exists a $q \in u$ and a $t \in P$ such that $t \leq p$ and $t \leq q$. Since $R``\{p\} \subseteq R``\{t\}$, $R``\{q\} \subseteq R``\{t\}$ we obtain

$$q \in u \to q \in R``\{p\} \to p \in R``\{q\} \to$$
$$p \in R``\{t\} \to t \in R``\{p\} \to t \in R``\{t\},$$

a contradiction.
Hence u is not dense in $(P/\sim, \leq)$. □

Boundedness Property

$M \subseteq N$ be an extension, κ regular cardinal in M.
The κ-**boundedness property** $Bd_{M,N}(\kappa)$:

$$(\forall x \subseteq M, x \in N)(\exists a \in M)(\exists y \in N)\,(|a|^M < \kappa \wedge y \subseteq a \wedge x = \bigcup y).$$

$Bd_M(\kappa)$ means $Bd_{M,V}(\kappa)$.

Lemma 8

Let $M \subseteq N$ be an extensions, k being cardinal in M. Then $Bd_{M,N}(\kappa)$ holds true if and only if

$$(\forall x \subseteq Ord, x \in N)(\exists a \in M)(\exists y \in N)\,(|a|^M < \kappa \wedge y \subseteq a \wedge x = \bigcup y).$$

Boundedness Property

If $V = M[G]$, where $G \subseteq P$ is a filter on a poset P generic over M, $|P|^M < \kappa$, then $Bd_M(\kappa)$.

If $x \subseteq b \in M$, $x \in M[G]$, then there exists a relation $r \subseteq P \times b$ such that $x = r''G$. Set

$$a = \{r''\{p\} : p \in P\}, \quad y = \{r''\{p\} : p \in G\}.$$

Theorem 9 (essentially P. Vopěnka 1972)

Let M be an inner model, κ being a cardinal in M. If $Bd_M(\kappa)$ holds true then there exists a partially ordered set $P \in M$, $|P|^M < \kappa$, and a filter $\sigma \subseteq P$ generic over M such that $V = M[\sigma]$.

Proof. We follow the proof in [VH], p. 207.
Let $\lambda = |[\kappa]^{<\kappa}|^V$. There exists a set $\rho \subseteq \lambda \times \kappa$

$$(\forall \tau \in [\kappa]^{<\kappa})(\exists \xi \in \lambda)\, \tau = \rho''\{\xi\}.$$

For such a ξ we set $r = \{((\xi, \eta), \eta) : \eta \in \kappa\}$ and we have $\tau = r''\rho$. Thus, $\mathrm{Dep}_M(\tau, \rho)$ for any $\tau \in [\kappa]^{<\kappa}$. By $Bd_M(\kappa)$ for any set $\mu \subseteq M$ there exists a set $a \in M$, $|a|^M < \kappa$ and a set $y \subseteq a$ such that $\mu = \bigcup y$. Therefore $\mathrm{Dep}_M(\mu, y)$. Since $|a|^M < \kappa$ there exists an injection $f : a \xrightarrow{1-1} \kappa$, $f \in M$. Then $f(y) \in [\kappa]^{<\kappa}$. Hence $\mathrm{Dep}_M(y, \rho)$. Therefore $\mathrm{Dep}_M(\mu, \rho)$, consequently ρ is a total support of V over M.

Using $Bd_M(\kappa)$ again, there exists a set $P \in M$, $|P|^M < \kappa$ and a set $\sigma \subseteq P$ such that $\rho = \bigcup \sigma$. Then $\mathrm{Dep}_M(\rho, \sigma)$. Thus σ is a total support of V over M. The Theorem follows by Theorem 5. □

Corollary 10

. Let $M_1 \subseteq M_2 \subseteq M_3$ be extensions, $M_1 \subseteq M_3$ being generic. Then both extensions $M_1 \subseteq M_2$ and $M_2 \subseteq M_3$ are generic as well.

Proof. If $M_1 \subseteq M_3$ is generic, then $Bd_{M_1,M_3}(\kappa)$for some κ. By Lemma 8 we have $Bd_{M_1,M_2}(\kappa)$ and $Bd_{M_2,M_3}(|\kappa|^{M_2})$. □

Approximation Property

The κ-**approximation condition** $Apr_{M,N}(\kappa)$[1]:

$$(\forall f \in N, f \text{ a function}, \mathrm{dom}(f) \in M, \mathrm{rng}(f) \subseteq M)$$
$$(\exists g : \mathrm{dom}(f) \longrightarrow M, g \in M)(\forall x \in \mathrm{dom}(f))$$
$$(|g(x)|^M < \kappa \wedge f(x) \in g(x)).$$

$Apr_M(\kappa)$ means $Apr_{M,V}(\kappa)$.

Theorem 11 (L.B. 1973)

Let M be an inner model. V is a κ-C.C. generic extension of M if and only if $Apr_M(\kappa)$ holds true.

"κ-C.C. generic" means "generic with a κ-C.C. cBa".

[1] In [FFS] the authors say that M κ-globally covers N. In [Sch] the author says that M uniformly κ-covers N. Both papers [FFS] and [Sch] contain an alternative proof of Theorem 11.

κ a measurable cardinal, U a κ-complete nonprincipal ultrafilter on κ.
By Mostowski collaps $\pi : {}^{\kappa}V/U \longrightarrow M$ the ultrapower ${}^{\kappa}V/U$ is isomorphic to an inner model M.
j is the elementary embedding $j : V \longrightarrow {}^{\kappa}V/U$.
$\pi(j(\kappa))$ is measurable in M. We can iterate.
Set $M_0 = M$, $j_0 = j$, $\pi_0 = \pi$. By induction
$M_{n+1} = M^{M_n} \subseteq M_n$, $j_{n+1} = j^{M_n}$, $\pi_{n+1} = \pi^{M_n}$.
We set $i_0 = j_0 \circ \pi_0$, $i_{n+1} = j_{n+1} \circ \pi_{n+1} \circ i_n$
M_ω is the direct limit of $\langle M_n, i_n : n \in \omega \rangle$.

An Application

Theorem 12 (L.B. 1973)

Let $N = \bigcap_n M_n$. *Then* $M_\omega \subseteq N$ *is a generic extension,* $cf(\kappa_\omega)^N = \omega$, $Card^N = Card^{M_\omega}$.

Idea of the Proof:

N is closed under Gödel operations.

N is almost universal since the cumulative hierarchy is $V_\xi^N = V_\xi \cap N \in N$.

$Fix = \{\xi \in Ord : i_0(\xi) = \xi\}$ is a proper class. For a given $x \in N$ we can construct $f : \mathrm{dom}(f) \xrightarrow[\text{onto}]{1-1} x$, $\mathrm{dom}(f) \subseteq Fix$, such that $i_n(f) = f$ for any $n \in \omega$.

We obtain AC in N.

An Application

Similarly we obtain $Apr_{M_\omega,N}(\kappa_\omega^+)$.
Since $i_0(\alpha) \in Card$ for any $\alpha \in Card$, we obtain $Card^N = Card^{M_\omega}$.

□

L.B. 1977 and independently P. Dehornoy 1978 have shown that the extension $M_\omega \subseteq N$ is actually the generic extension over Príkry forcing constructed in M_ω.

We present some results about existence of specific extensions of an inner model.

Borel denotes the set of all Borel subsets of $[0,1]$.

M is an inner model. There is a mapping

$$\# : \mathit{Borel}^M \longrightarrow \mathit{Borel},$$

preserving complements and countable unions of sets from M, see R.M. Solovay 1970, T. Jech [J] p. 511.

A real $c \in [0,1]$ is **Cohen over the inner model** M if $c \in \#(A)$ for any comeager Borel set $A \subseteq [0,1]$, $A \in M$.

Similarly a real **random over** M.

Theorem 13 (Two Cohen's Theorem)

Let M be an inner model, $|\mathcal{P}(\omega) \cap M| = \aleph_0$, $a \in [0,1]$ being a real. Then there exist two reals c_1, c_2 Cohen over M such that $c_1 + c_2 = a$.

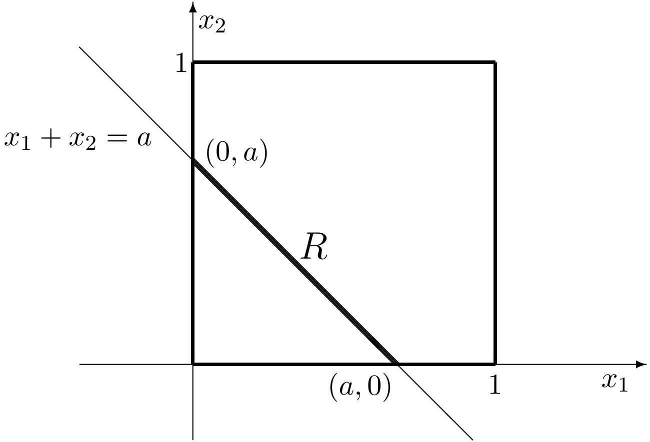
R. Solovay 1960's.

B. Balcar and P. Hájek 1970's.

J. Truss 1970's.

Proof was not published. I suppose that the first proof (different from those by the above mentioned authors) was published in my paper *Cogeneric extensions*, in Logic Colloquium '77.

Proof. Let $C = \{c \in [0,1] : c$ is Cohen over $M\}$.
Since $|\mathcal{P}(\omega) \cap M| = \aleph_0$, also $|Borel^M| = \aleph_0$.
Thus C is comeager G_δ-set.



Proof of Two Cohen's Theorem

The straight line $x_1 + x_2 = a$ intersects the square $[0,1] \times [0,1]$ in a segment R with end-points $(0, a)$ and $(a, 0)$. Denote

$$C_1 = C \times [0,1], \quad C_2 = [0,1] \times C.$$

The projections from the axes x_1, x_2 on the straight line $x_1 + x_2 = a$ are homeomorphisms. Thus both sets $C_1 \cap R$ and $C_2 \cap R$ are comeager G_δ-sets in R. Hence $C_1 \cap C_2 \cap R$ is comeager in R as well; therefore non-empty. If $(c_1, c_2) \in C_1 \cap C_2 \cap R$ then $c_1, c_2 \in C$ are Cohen over M and $c_1 + c_2 = a$. □

Similar arguments work for random reals.

Two Extensions of an Inner Model

Corollary 14

Let M *be an inner model,* $|\mathcal{P}(\omega) \cap M| = \aleph_0$*. For any real* r *random over* M*, there exist two reals* c_1, c_2 *Cohen over* M *such that* $c_1 + c_2 = r$*.*

Corollary 15 (Non-cogeneric Extensions)

Let M *be an inner model,* $|\mathcal{P}(\omega) \cap M| = \aleph_0$*. Assume that* $a \in [0, 1]$ *is a real that cannot be in a generic extension of* M $($*e.g.,* $M = L$ *and* $a = 0^{\#})$*. Then there exist two reals* c_1, c_2 *Cohen over* M *such that there is no generic extension* $M \subseteq N$ *which would be a common extension of both inner models* $M[c_1], M[c_2]$*.*

Let M be a transitive set, $(M, \in)$ being a model of ZFC. $M \subseteq N$ is an extension of M if N is a transitive set, model of ZFC, and $Ord^M = Ord^N$.

Corollary 16 (Non-coextensive Extensions)

If M is a countable transitive model of ZFC, then there exist two reals c_1, c_2 Cohen over M such that there is no common extension of both models $M[c_1], M[c_2]$.

Proof. Since M is countable there is a real a that codes the well ordered set $(Ord^M, \in)$. Take $c_1 + c_2 = a$. □

The extensions $M \subseteq N_1$ and $M \subseteq N_2$ are **disjoint** if $N_1 \cap N_2 = M$.

The extensions $M \subseteq N_1$ and $M \subseteq N_2$ are **separated** if for any disjoint sets $x_1, x_2 \subseteq M$, $x_1 \in N_1$, $x_2 \in N_2$ there exists a set $x \in M$ such that $x_1 \subseteq x$ and $x_2 \cap x = \emptyset$, equivalently, if $x_1, x_2 \subseteq a \in M$, then $x_1 \subseteq x$ and $x_2 \subseteq a \setminus x$. Thus, the property "separated" is symmetric.

The extensions $M \subseteq N_1$ and $M \subseteq N_2$ are **cogeneric** if there exists a generic extension $M \subseteq N$ such that $N_1 \subseteq N$ and $N_1 \subseteq N$.

Two Extensions of an Inner Model

Theorem 17 (L.B. [Bu4])

Generic extensions. $M \subseteq N_1$ *and* $M \subseteq N_2$ *are separated if and only if there exist posets* $P_i \in M$, $i = 1, 2$ *and a filter* $G \subseteq P_1 \times P_2$ *generic over* M *such that* $N_i = M[G \cap P_i]$, $i = 1, 2$.

Proof. Let $M \subseteq N_1$ and $M \subseteq N_2$ be generic separated extensions. $P_i \in M$ is poset, $G_i \subseteq P_i$ is filter generic over M and $N_i = M[G_i]$ for $i = 1, 2$. G_1 is generic over N_2: if $A \subseteq P_1$, $A \in N_2$, $A \cap G_1 = \emptyset$, then there exists $Y \in M$ such that $A \subseteq Y$ and $Y \cap G_1 = \emptyset$. Then Y is not dense in P_1. So neither A is dense in P_1. □

Corollary 18 (L.B. [Bu4])

Separated generic extensions are cogeneric.

Theorem 19 (L.B. [Bu4])

Separated extensions are disjoint.

Proof. Let $M \subseteq N_1$ and $M \subseteq N_2$ be separated extensions. Assume that the set $x \subseteq a \in M$ belongs both to N_1 and N_2. We show that $x \in M$. Indeed, since the extensions are separated, there exists a set $z \in M$ such that $x \subseteq z$ and $(a \setminus x) \cap z = \emptyset$. Then $z = x$. Hence $x \in M$. □

Two Extensions of an Inner Model

Theorem 20 (L.B. [Bu4])

It is consistent with **ZFC** *that there exist cogeneric disjoint extensions that are not separated.*

Proof. Let $|\mathcal{P}(\omega) \cap M| = \aleph_0$. There exist a real $r \subseteq \omega$ random over M and a real $c \subseteq \omega$ Cohen over $M[r]$. If $f : \omega \longrightarrow (\omega \setminus r)$, $f \in M[r]$ is a bijection, then $c_0 = f_{-1}((\omega \setminus r) \cap c)$ is Cohen over $M[r]$ and $M[r][c_0] \subseteq M[r][(\omega \setminus r) \cap c]$.
The extensions $M \subseteq M[r]$ and $M \subseteq M[c_0]$ are cogeneric and disjoint. If $r \subseteq x \in M$, then $\omega \setminus x$ is finite. Thus $c_0 \nsubseteq \omega \setminus x$. Therefore the extensions $M \subseteq M[r]$, $M \subseteq M[c_0]$ are not separated. □

Theorem 21 (L.B. [Bu4])

Let $M \subseteq N_1$, $M \subseteq N_2$ *be extensions. Then* N_1, N_2 *are disjoint cogeneric extensions if and only if there exist Boolean algebras* $B_1, B_2 \subseteq B \in M$ *complete in* M *such that for every* $u \in B$, $u \neq 0$ *there exists a* $v \leq u$, $v \neq 0$ *such that* $B_1|v \cap B_2|v = \{0, v\}$ *and an ultrafilter* $G \subseteq B$ *generic over* M *such that* $N_i = M[G \cap B_i]$ *for* $i = 1, 2$.

Thank you

for your attention

Approximation Property – Proof

Lemma 22 (L.B. [Bu1])

Let M be an inner model. If $a \subseteq \omega_0$ and $Apr_M(\aleph_1)$ holds true, then $M[a]$ is a generic extension of M.

Proof. We consider the set a as an element of $[0, 1]$ and we set

$$j = \{A \in \mathit{Borel}^M : a \in \#(A)\}.$$

j is an ultrafilter on Borel^M closed under intersections of countable families from M and $M[a] = M[j]$. We show that j is a support.

Claim: $(\forall r, r \in M, r$ relation, $\mathrm{dom}(r) \subseteq \mathit{Borel}^M)$

$$(\exists h \in M \text{ function})\, r''j = h_{-1}(j).$$

Approximation Property – Proof

Let $r \subseteq \mathit{Borel}^M \times P$, where $P \in M$. For $x \in P$ set

$$R(x) = \{A \in \mathit{Borel}^M : (A, x) \in r\}.$$

Evidently $R \in M$. By **AC** in $M[a]$ there exists a function $f : P \longrightarrow \mathit{Borel}^M$ such that $f(x) \in j \cap R(x)$, if $j \cap R(x) \neq \emptyset$ and $f(x) = \emptyset \in \mathcal{B}^M$ otherwise.

By $Apr_M(\aleph_1)$ there exists a function $g \in M$, $g : P \longrightarrow ([\mathit{Borel}]^{<\aleph_1})^M$, such that $f(x) \in g(x)$ for each $x \in P$. We set $h(x) = \bigcup(g(x) \cap R(x))$. Then $\mathrm{rng}(h) \subseteq \mathit{Borel}^M$, $h \in M$ and

$$h(x) \in j \equiv j \cap R(x) \neq \emptyset.$$

Thus $r''j = h_{-1}(j)$.

If $\mathrm{Dep}_M(y_i, j)$, then $y_i = (h_i)_{-1}(j)$, set

$$h(x) = \begin{cases} h_1(x) \setminus h_2(x) & \text{if } x \in \mathrm{dom}(h_1) \cap \mathrm{dom}(h_2), \\ h_1(x) & \text{if } x \in \mathrm{dom}(h_1) \setminus \mathrm{dom}(h_2). \end{cases}$$

Then $h \in M$ and $y_1 \setminus y_2 = h_{-1}(j)$. □

Corollary 23 (L.B. [Bu5] and J.L. Krivine [K])

If $Apr_M(\kappa)$ and $\rho \subseteq P \in M$, then the inner model $M[\rho]$ is a generic extension of M.

Proof. Collapse a cardinal $\lambda \geq \kappa$, $\lambda \geq |P|$ to $\aleph_0$. The collapse map and ρ may be coded by a real a. Then $Apr_{M,V[a]}(\aleph_1)$ holds true. Working in $V[a]$ apply the Lemma 22. □

Lemma 24

If V is a generic extension of M and $Apr_M(\kappa)$ holds true, then V is a κ-C.C. generic extension of M.

Proof. Assume that $V = M[G]$, G is an ultrafilter on an M-complete Boolean algebra B generic over M.
Let $\mathcal{D} = \{D \subseteq B : D \text{ is a partition of B} \wedge D \in M\}$.
We set $f(D) = a \in G \cap D$ for $D \in \mathcal{D}$. By $Apr_M(\kappa)$ there exists $g : \mathcal{D} \longrightarrow [B]^{<\kappa}$, such that $g \in M$ and $f(D) \in g(D)$ for each $D \in \mathcal{D}$.
Then $a = \bigwedge_{D \in \mathcal{D}} \bigvee g(D) \in G$ and the Boolean algebra $B|a$ is κ-C.C. □

Approximation Property – Proof

The Theorem 11 follows from the following

Lemma 25

If B is a complete atomless κ-C.C. Boolean algebra, then the first cardinal λ such that B is not (λ, κ)-distributive is $\lambda \leq \kappa$.
Thus if $Apr_{M,N}(\kappa)$ holds true, then $N = M[\rho]$, where $\lambda = |\mathcal{P}(\kappa)^N|^N$ and $\rho \subseteq \lambda \times \kappa$ is such that

$$\mathcal{P}(\kappa)^N = \{\rho''\{\xi\} : \xi \in \lambda\}.$$

Proof. If $\lambda > \kappa$ we can construct a strictly decreasing κ-chain. Thus $\lambda \leq \kappa$.

Approximation Property – Proof

Let $M \subseteq N$, ρ be as in the Lemma. Assume that $M[\rho] \neq N$. Then there exists a set of ordinals $a \subseteq Ord$, $a \in N$ such that $a \notin M[\rho]$.

Since $Apr_{M[\rho],N}(\kappa)$ holds true, by Corollary 23, $M[\rho][a]$ is a generic extension of $M[\rho]$. Therefore there exist a κ-C.C. Boolean algebra B and an ultrafilter $G \subseteq B$ generic over $M[\rho]$ such that $M[\rho][a] = M[\rho][G]$. Since $\mathcal{P}(\kappa)^N \subseteq M[\rho][a]$, we can assume that the Boolean algebra B is (κ, κ)-distributive. Thus $\lambda > \kappa$. However, since $a \notin M[\rho]$, we can assume that the cBa B is atomless Therefore $\lambda \leq \kappa$ – a contradiction. □

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