Preserving levels of projective determinacy and regularity properties

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Overview

- We study the preservation of levels of projective determinacy and regularity properties under iterations of 'simply' definable forcing notions.
- This is done by using a technique called capturing.
- All the well-known tree forcings are 'simply' definable, hence our results are applicable to the study of cardinal characteristics.
- The results are from a joint project with Jonathan Schilhan and Philipp Schlicht.
- This project is a sequel to the paper 'Preserving levels of projective determinacy by tree forcings' by F. Castiblanco and P. Schlicht.

Preserving ‘Every real has a sharp’

Sharps

Definition

$0^\sharp$ exists iff each (at least one) of the following objects exist:

1. An uncountable set of ordinals which are order-indiscernible over L.
2. A non-trivial, elementary embedding $j : L \to L$.
3. A well-founded, remarkable Ehrenfeucht-Mostowski type.
4. A countable structure $(L_\alpha, \in, U)$ such that
   - $(L_\alpha, \in)$ is a model of ZFC^- with a largest cardinal κ,
   - $(L_\alpha, \in, U)$ is a model of Σ_0-separation,
   - U is a $<\kappa$-complete ultrafilter on $\mathcal{P}(\kappa)^{L_\alpha}$ and
   - all iterated ultrapowers of $(L_\alpha, \in, U)$ are wellfounded.

More generally, $x^\sharp$ is defined for any real $x \in \omega^\omega$ by replacing L with $L[x]$.

The existence of $x^\sharp$ follows from the existence of a measurable cardinal.

We say that 'Every real has a sharp' iff $\forall x \in \omega^\omega : x^\sharp$ exists.

Preserving Large Cardinals

The following large cardinal preservation theorem is well known:

Theorem (Laver)

Let κ be supercompact and let V be suitably prepared. Then the supercompactness of κ is indestructible by any $<\kappa$-directed closed forcing notion.

Maybe less known:

Theorem (Johnstone)

Let κ be strongly unfoldable and let V be suitably prepared. Then the strong unfoldability of κ is indestructible by any $<\kappa$-closed, κ^+-c.c. (κ-proper) forcing notion.

Question: What are other examples of large cardinals where such an 'exact' preservation can be shown for a larger class of forcing notions?

Determinacy

Let A be a subset of 2^ω. In the game $G(A)$, two players play $n_i \in \{0,1\}$ in turn. Player I wins iff $\vec{n} = \langle n_i \mid i \in \omega \rangle \in A$.

Table: $G(A)$

	Round 0		Round 1	
Player I	n_0		n_2	
Player II		n_1		n_3

We call A determined iff one of the players has a winning strategy in the game $G(A)$. We say that $\mathbf{\Pi}^1_1$-determinacy holds iff every (co-)analytic set is determined.

Theorem (Harrington (1978), Martin (1970))

The following statements are equivalent:

- $\mathbf{\Pi}^1_1$*-determinacy holds.*
- *Every real has a sharp.*

Capturing

But how can the statement ‘Every real has a sharp’ be preserved?

The answer lies in the technique of capturing:

Definition

Let $\mathbb{P}$ be a forcing notion. We say that $\mathbb{P}$ is captured iff

$$\forall p \in \mathbb{P}\ \forall\ \mathbb{P}\text{-names } \dot{\tau} \text{ for a real } \forall y \in \omega^\omega \exists z \in \omega^\omega\ \exists \mathbb{Q} \in L[y,z]\ \exists q \leq_{\mathbb{P}} p\colon$$

$$q \Vdash_{\mathbb{P}} \exists H\colon H \text{ is } (L[y,z], \mathbb{Q})\text{-generic} \wedge \dot{\tau} \in L[y,z][H]$$

Preserving ‘Every real has a sharp’

Theorem

Assume that $V \models$ 'Every real has a sharp' and let $\mathbb{P}$ be a forcing notion. If $\mathbb{P}$ is captured, then $V^{\mathbb{P}} \models$ 'Every real has a sharp'.

Proof.

Working in V let $p \in \mathbb{P}$ and $\dot{\tau}$ a $\mathbb{P}$-name for a real be arbitrary. Since $\mathbb{P}$ is captured, there now exist $q \leq_{\mathbb{P}} p$, $z \in \omega^\omega$ and $\mathbb{Q} \in L[z]$ such that
$q \Vdash_{\mathbb{P}} \exists H\colon H$ is $(L[z], \mathbb{Q})$-generic $\wedge\ \dot{\tau} \in L[z][H]$.
Since $z^\sharp$ exists, there is a non-trivial, elementary embedding
$j : L[z] \to L[z]$ with $\mathrm{crit}(j) > |\mathbb{Q}|$. Hence, j can be lifted to
$j^* : L[z]^{\mathbb{Q}} \to L[z]^{\mathbb{Q}}$, and we can conclude that $q \Vdash_{\mathbb{P}} \exists H\ \exists j^* : \dot{\tau} \in L[z][H] \wedge$
$j^* : L[z][H] \to L[z][H]$ is a non-trivial, elementary embedding.
In particular, $q \Vdash_{\mathbb{P}} \exists \tilde{j}\ \ \tilde{j} : L[\dot{\tau}] \to L[\dot{\tau}]$ is a non-trivial, elementary embedding, hence $q \Vdash_{\mathbb{P}} \dot{\tau}^\sharp$ exists. □

Variants of Capturing 1

The following is a strengthening of capturing:

Definition

Let $\mathbb{P}$ and $\mathbb{Q}$ be forcing notions and such that $\mathbb{Q}$ is definable. We say that $\mathbb{Q}$ captures $\mathbb{P}$ iff

$$\forall p \in \mathbb{P} \ \forall \ \mathbb{P}\text{-names } \dot{\tau} \text{ for a real } \forall y \in \omega^\omega \ \exists z \in \omega^\omega \ \exists q \leq_{\mathbb{P}} p \colon$$

$$q \Vdash_{\mathbb{P}} \exists H \colon H \text{ is } (L[y,z], \mathbb{Q}^{L[y,z]})\text{-generic} \wedge \dot{\tau} \in L[y,z][H]$$

Variants of Capturing 2

The following is a strengthening of $\mathbb{Q}$ captures $\mathbb{P}$:

Definition

Let $\mathbb{P}$ and $\mathbb{Q}$ be forcing notions and such that $\mathbb{Q}$ is definable. We say that $\mathbb{Q}$ uniformly captures $\mathbb{P}$ iff

$$\forall p \in \mathbb{P} \; \forall \; \mathbb{P}\text{-names } \dot{\tau} \text{ for a real } \exists z \in \omega \; \exists \; \mathbb{P}\text{-name } \dot{H} \; \forall y \in \omega^\omega \; \exists q \leq_{\mathbb{P}} p :$$

$$q \Vdash_{\mathbb{P}} \dot{H} \text{ is } (L[y,z], \mathbb{Q}^{L[y,z]})\text{-generic} \wedge \dot{\tau} \in L[z][\dot{H}]$$

Examples

Lemma (Castiblanco - Schlicht)

If ω_1 is inaccessible to the reals, then:

- *Cohen forcing uniformly captures Sacks and Silver forcing.*
- *Mathias forcing uniformly captures Laver, Mathias and Miller forcing.*

Lemma

If $BP(\boldsymbol{\Delta}^1_2)$ holds, then Cohen forcing uniformly captures Sacks and Silver forcing.

Lemma

If $BP(\boldsymbol{\Sigma}^1_2)$ holds, then Cohen forcing uniformly captures Miller forcing.

Lemma (Schilhan)

Let $\mathbb{P}$ be a countable support iteration of Sacks or Silver forcing. If $BP(\boldsymbol{\Delta}^1_2)$ holds, then $\mathbb{P}$ is captured.

Suslin Forcing

We will need the following definitions:

Definition

Let $\mathbb{P} = (\text{dom}(\mathbb{P}), \leq_{\mathbb{P}})$ be a forcing notion such that $\text{dom}(\mathbb{P}) \subseteq \omega^\omega$. We say that $\mathbb{P}$ is Suslin iff $\text{dom}(\mathbb{P})$ and $\leq_{\mathbb{P}}$ have $\mathbf{\Sigma}^1_1$ definitions.
We say that $\mathbb{P}$ is strongly Suslin iff additionally the incompatibility relation $\perp_{\mathbb{P}}$ also has a $\mathbf{\Sigma}^1_1$ definition.

and

Definition

Let $\mathbb{P}$ be a Suslin forcing. We say that $\mathbb{P}$ is proper-for-candidates iff for every countable, transitive model N containing the real parameters for the Suslin definitions of $\text{dom}(\mathbb{P})$ and $\leq_{\mathbb{P}}$ and satisfying ZFC*, and every $p \in \mathbb{P}^N$ there exists $q \in \mathbb{P}$ such that $q \leq_{\mathbb{P}} p$ and q is $(N, \mathbb{P})$-generic.

Capturing of Iterations

We can now state our main theorem:

Theorem (Sch.-Sch.-Sch.)

Let $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{P}_\beta \colon \alpha \leq \kappa, \beta < \kappa \rangle$ be a countable support iteration of Suslin forcing notions $\dot{P}_\beta$ such that for every $\alpha < \kappa$ we have:

$$\Vdash_{\mathbb{P}_\alpha} \forall A \in [\omega^\omega]^\omega \colon \dot{P}_\alpha \in L[A] \Rightarrow L[A] \vDash \dot{P}_\alpha \text{ is proper-for-candidates.}$$

If ω_1 is inaccessible to the reals, then $\mathbb{P}$ is captured.

Sketch of Proof.

For simplicity let us assume that $\mathbb{P}$ is an iteration of Sacks forcing. Let $p \in \mathbb{P}$, $\dot{\tau}$ a $\mathbb{P}$-name for a real and $y \in \omega^\omega$ be arbitrary. Let $(\dot{s}_\beta)_{\beta<\kappa}$ be a $\mathbb{P}$-name for the sequence of generic Sacks reals.
Using continuous reading of names we can assume that there exists $\tilde{u} \subseteq \kappa$ countable and a continuous function $\tilde{f} : (2^\omega)^{\tilde{u}} \to \omega^\omega$ such that w.l.o.g. $p \Vdash_{\mathbb{P}} \dot{\tau} = \tilde{f}((\dot{s}_\beta)_{\beta\in\tilde{u}})$. □

Sketch of Proof (Cont.)

Furthermore, we can assume w.l.o.g. that there exists $(u_\alpha)_{\alpha\in\mathrm{supp}(p)} \subseteq [\kappa]^\omega$ and $(f_\alpha)_{\alpha\in\mathrm{supp}(p)}$ with $f_\alpha : (2^\omega)^{u_\alpha} \to \mathcal{P}(2^{<\omega})$ continuous such that $\forall\alpha \in \mathrm{supp}(p)\colon p \restriction \alpha \Vdash_{\mathbb{P}_\alpha} \dot{p}(\alpha) = f_\alpha((\dot{s}_\beta)_{\beta\in u_\alpha})$. Set $u^* := \tilde{u} \cup \bigcup_{\alpha\in\mathrm{supp}(p)} u_\alpha$ and let $\mathrm{mos} : u^* \to \alpha^*$ denote the Mostowski collapse of u^*. Let $\pi : \alpha^* \to u^*$ denote the uncollapse and set $\pi(\alpha^*) := \kappa$. Now code the 'transitive collapse' of u^*, $\tilde{u}$, $\tilde{f}$, $(u_\alpha)_{\alpha\in\mathrm{supp}(p)}$ and $(f_\alpha)_{\alpha\in\mathrm{supp}(p)}$ as $z \in \omega^\omega$.

Let $\mathbb{Q} = \langle \mathbb{Q}_\alpha, \dot{Q}_\beta\colon \alpha \leq \alpha^*, \beta < \alpha^* \rangle$ be a countable (full) support iteration of Sacks forcing of length α^* in $L[y,z]$. We will show that there exists a $\mathbb{P}$-name $\dot{H}$ and a condition $p^* \leq_\mathbb{P} p$ such that $p^* \Vdash_\mathbb{P} \dot{H}$ is $(L[y,z],\mathbb{Q})$-generic $\wedge\ \dot{\tau} \in L[y,z][\dot{H}]$. □

Capturing of Iterations

Sketch of Proof (Cont.)

To this end we define by induction on $\alpha \leq \alpha^*$ an embedding $i_\alpha : \mathbb{Q}_\alpha \to \mathbb{P}_{\pi(\alpha)}$, i.e. for every $q_1, q_2 \in \mathbb{Q}$ we have $q_1 \leq_{\mathbb{Q}} q_2$ iff $i_\alpha(q_1) \leq_{\mathbb{P}_{\pi(\alpha)}} i_\alpha(q_2)$, with $\text{supp}(i_\alpha(q)) = \pi[\text{supp}(q)]$ for every $q \in \mathbb{Q}$, and simultaneously we show using a preservation-of-properness argument that for every $q \in \mathbb{Q}_\alpha$ there exists a $p' \leq_{\mathbb{P}} i_\alpha(q)$ such that $p' \Vdash_{\mathbb{P}_{\pi(\alpha)}} i_\alpha^{-1}[\dot{G}_{\mathbb{P}_{\pi(\alpha)}}]$ is $(L[y,z], \mathbb{Q}_\alpha)$-generic.
Since the 'transitive collapse' of $(u_\alpha)_{\alpha \in \text{supp}(p)}$ and $(f_\alpha)_{\alpha \in \text{supp}(p)}$ belong to $L[y,z]$, there exists a $q \in \mathbb{Q}$ such that $i_{\alpha^*}(q) = p$ (in the sense that $i_{\alpha^*}(q) \leq_{\mathbb{P}} p$ and $p \leq_{\mathbb{P}} i_{\alpha^*}(q)$). Hence we can deduce that there exists a $p^* \leq_{\mathbb{P}} p$ such that $p^* \Vdash_{\mathbb{P}} i_{\alpha^*}^{-1}[\dot{G}_{\mathbb{P}}]$ is $(L[y,z], \mathbb{Q})$-generic.
Since the 'transitive collapse' of $\tilde{u}$ and $\tilde{f}$ belong to $L[y,z]$, we have $p^* \Vdash_{\mathbb{P}} \dot{\tau} = \tilde{f}((\dot{s}_{\pi(\beta)})_{\beta \in \text{mos}[\tilde{u}]} \in L[y,z][i_{\alpha^*}^{-1}[\dot{G}_{\mathbb{P}}]]$. Hence, if we set $\dot{H} := i_{\alpha^*}^{-1}[\dot{G}_{\mathbb{P}}]$ then $p^* \Vdash_{\mathbb{P}} \dot{H}$ is $(L[y,z], \mathbb{Q})$-generic $\wedge\ \dot{\tau} \in L[y,z][\dot{H}]$. □

Preserving Regularity Properties

The Baire Property

Let $\mathcal{M}$ denote the Borel ideal of all meager sets of 2^ω.

Definition

We say that $\text{BP}(\boldsymbol{\Delta}^1_2)$ holds iff every $\boldsymbol{\Delta}^1_2$ set $X \subseteq \omega^\omega$ has the Baire Property, i.e. there exists $O \subseteq \omega^\omega$ open such that $X \triangle O$ is meager.

Similarly, we define when $\text{BP}(\boldsymbol{\Sigma}^1_2)$ holds.

And recall:

Theorem (Judah-Shelah (1989), Solovay (1969))

$BP(\boldsymbol{\Delta}^1_2)$ holds iff for $\forall x \in \omega^\omega : \bigcup(\mathcal{M} \cap L[x]) \neq 2^\omega$, i.e. there exists a Cohen real over $L[x]$.

$BP(\boldsymbol{\Sigma}^1_2)$ holds iff for $\forall x \in \omega^\omega : \bigcup(\mathcal{M} \cap L[x]) \in \mathcal{M}$, i.e. there exists a comeager set of Cohen reals over $L[x]$.

Preserving Regularity Properties 1

Theorem (Sch.-Sch.-Sch.)

Assume that $V \vDash BP(\mathbf{\Delta}^1_2)$ *and let* $\mathbb{P}$ *be a forcing notion. If Cohen forcing uniformly captures* $\mathbb{P}$*, then* $V^{\mathbb{P}} \vDash BP(\mathbf{\Delta}^1_2)$*.*

Proof.

We will show that in $V^{\mathbb{P}}$ there exists a Cohen real over $L[x]$ for every real $x \in \omega^\omega$. Note that if c is a Cohen real over $L[x, y]$, then it is also a Cohen real over $L[x]$.

Working in V assume that $p \in \mathbb{P}$ and $\dot\tau$ is a $\mathbb{P}$ name for a real. By uniform capturing, there exist $z \in \omega^\omega$ and a $\mathbb{P}$-name $\dot c$ such that for every $y \in \omega^\omega$ there is a $q \leq_{\mathbb{P}} p$ with

$$q \Vdash_{\mathbb{P}} \dot c \text{ is a Cohen real over } L[y, z] \text{ and } \dot\tau \in L[z][\dot c].$$

Let $c_0 \in \omega^\omega$ be a Cohen real over $L[z]$, which exists since $\mathrm{BP}(\mathbf{\Delta}^1_2)$ holds. Set $y := c_0$ and pick a corresponding condition $q \leq_{\mathbb{P}} p$ with the required properties.

By mutual genericity we have $q \Vdash_{\mathbb{P}} c_0$ is a Cohen real over $L[z][\dot c] \supseteq L[\dot\tau]$. □

Preserving Regularity Properties 2

Theorem (Sch.-Sch.-Sch.)

Assume that $V \models BP(\mathbf{\Sigma}^1_2)$ and let $\mathbb{P}$ be a forcing notion. If Cohen forcing uniformly captures $\mathbb{P}$, then $V^{\mathbb{P}} \models BP(\mathbf{\Sigma}^1_2)$.

Proof.

We will show that in $V^{\mathbb{P}}$ the set $\bigcup(\mathcal{M} \cap L[x])$ is meager for every real $x \in \omega^\omega$. Working in V let $p \in \mathbb{P}$ and $\dot{\tau}$ be a $\mathbb{P}$-name for a real. Again, by uniform capturing, there exist $z \in \omega^\omega$ and a $\mathbb{P}$-name $\dot{c}$ with the required properties.
Let $\mathcal{M}(2^\omega \times 2^\omega)$ denote the Borel ideal of all meager sets of $2^\omega \times 2^\omega$. By assumption, there exists an $B \in \mathcal{M}(2^\omega \times 2^\omega) \cap V$ such that $\bigcup(\mathcal{M}(2^\omega \times 2^\omega) \cap L[z]) \subseteq B$. Let B be coded by $y \in \omega^\omega$. Then there exists $q \leq_{\mathbb{P}} p$ such that

$$q \Vdash_{\mathbb{P}} \dot{c} \text{ is a Cohen real over } L[y, z] \text{ and } \dot{\tau} \in L[z][\dot{c}].$$

□

Preserving Regularity Properties 2

Proof (Cont.)

Let G be $(V, \mathbb{P})$-generic and working in $V[G]$ set $X := \{u \in 2^\omega : (\dot{c}^G, u) \in B\}$. We claim that X is meager and contains every meager set coded in $L[\dot{\tau}^G]$.
To see that X is meager, recall that by the Kuratowski-Ulam Theorem there exists a comeager set $C \subseteq 2^\omega$ coded in $L[y, z]$ such that for every $x \in C$ the set $\{u \in 2^\omega : (x, u) \in B\}$ is meager. Since $\dot{c}^G$ is a Cohen real over $L[y, z]$, we have $\dot{c}^G \in C$. Hence, X is indeed meager.
Now assume that Y is a Borel meager set coded in $L[z][\dot{c}^G] \supseteq L[\dot{\tau}^G]$. Since $\dot{c}^G$ is also a Cohen real over $L[z]$, there exists a $B' \in \mathcal{M}(2^\omega \times 2^\omega) \cap L[z]$ such that $Y = \{u \in 2^\omega : (\dot{c}^G, u) \in B'\}$. Since we have $B' \subseteq B$ in V as well as in $V[G]$ by absoluteness, it follows that $Y \subseteq X$ holds in $V[G]$. □

Destroying Regularity Properties

Theorem (Sch.-Sch.-Sch.)

Let $\mathbb{MI}$ denote Miller forcing and assume $V = L(Add(\omega, \omega_1))$. Then $V^{\mathbb{MI}} \models \neg BP(\mathbf{\Delta}^1_2)$.

Proof (of Thm.)

Working in V, we assume towards a contradiction that

$$p \Vdash_{\mathbb{MI}} \dot{c} \in \omega^\omega \text{ is a Cohen real over } L[\dot{x}_{\text{gen}}]$$

for some $p \in \mathbb{MI}$ and an $\mathbb{MI}$-name $\dot{c}$. Using continuous reading of names we may assume that $f : [p] \to \omega^\omega$ is continuous and $p \Vdash f(\dot{x}_{\text{gen}}) = \dot{c}$. □

Claim

There exists $q \leq_{\mathbb{MI}} p$ such that $f(x)$ is a Cohen real over $L[x]$ for every $x \in [q]$.

Destroying Regularity Properties

Proof (of Claim).

For every $\alpha < \omega_1$ the set

$$B_\alpha := \{(x, z) \in (\omega^\omega)^2 \colon z \in \bigcup(\mathcal{M} \cap L_\alpha[x])\}$$

is a $\Delta^1_1(y)$ set, where $y \in \omega^\omega$ is a real coding α. In particular, B_α is coded in L for every $\alpha < \omega_1$, since $\omega_1^L = \omega_1$. Now note that for every $\alpha < \omega_1$ the set $X_\alpha := \{x \in [p] : (x, f(x)) \in B_\alpha\}$ is bounded and coded in $L[p, f]$:
If it were not bounded, then (by a result of Kechris) it would contain the branches of a superperfect tree $r \leq_{\mathbb{MI}} p$. But then $r \Vdash_{\mathbb{MI}} f(\dot{x}_{\text{gen}})$ is not a Cohen real over $L[\dot{x}_{\text{gen}}]$, since '$\forall x \in [r] \colon (x, f(x)) \in B_\alpha$' is $\mathbf{\Pi}^1_1$ and therefore absolute. □

Proof (of Claim) (Cont.)

Let $\eta\colon \omega^\omega \to [p]$ be the canonical homeomorphism, and note that $\eta^{-1}[X_\alpha]$ is bounded as well. The statement '$\eta^{-1}[X_\alpha]$ is bounded' is $\Sigma^1_2(p, f)$ and therefore absolute between $L[p, f]$ and V.
Since there exists a Cohen real over $L[p, f]$, there is an unbounded real d over $L[p, f]$. In particular, d is unbounded over $\eta^{-1}[X_\alpha]$ for every $\alpha < \omega_1$. Now we pick $q \leq_{\mathbb{MI}} p$ such that $d \leq^* \eta^{-1}(x)$ for every $x \in [q]$. But then q is as desired. □

Proof (of Thm.) (Cont.)

Now consider the set $A := \{f(x) + x : x \in [q]\}$. We note that A is a set of Cohen reals over L, since for any $x \in [q]$ we have that $f(x) + x$ is a translate of the Cohen real $f(x)$ over $L[x]$, and thus again Cohen over $L[x]$.
Moreover, $A \subseteq \omega^\omega$ is analytic and unbounded, and therefore contains the branches of a superperfect tree T. Then $[T]$ is a superperfect set of Cohen reals over L. However, by a result of Spinas, this is impossible in $L(Add(\omega, \omega_1))$. □

$\mathbf{\Delta}^1_3$ Relations

Generic Absoluteness

Lemma

Assume that $V \vDash$ 'Every real has a sharp' and let $\mathbb{P}$ be a forcing notion. If $\mathbb{P}$ is captured by forcing notions of size $< \omega_1^V$, then $V \prec_{\mathbf{\Sigma}^1_3} V^{\mathbb{P}}$.

Recall: Capturing

Let $\mathbb{P}$ be a forcing notion. We say that $\mathbb{P}$ is captured by forcing notions with property φ iff

$$\forall p \in \mathbb{P}\ \forall\ \mathbb{P}\text{-names } \dot{\tau} \text{ for a real } \forall y \in \omega^\omega \exists z \in \omega^\omega\ \exists \mathbb{Q} \in L[y,z]\ \exists q \leq_{\mathbb{P}} p:$$

$$L[y,z] \vDash \varphi(\mathbb{Q}) \wedge q \Vdash_{\mathbb{P}} \exists H\colon H \text{ is } (L[y,z],\mathbb{Q})\text{-generic} \wedge \dot{\tau} \in L[y,z][H]$$

Proof (of Lemma).

Let $\varphi(x)$ be a $\mathbf{\Sigma}^1_3$-formula and let $\psi(x,y)$ be a $\mathbf{\Pi}^1_2$-formula such that $\varphi(x) = \exists y\ \psi(x,y)$. Let $a \in \omega^\omega \cap V$ and assume that $V^{\mathbb{P}} \vDash \varphi(a)$. Hence there exists $b \in \omega^\omega \cap V^{\mathbb{P}}$ with $V^{\mathbb{P}} \vDash \psi(a,b)$.
Since $\mathbb{P}$ is captured by forcing notions of size $< \omega_1^V$, there exist $z \in \omega^\omega \cap V$, $\mathbb{Q} \in L[a,z]$ with $|\mathbb{Q}| < \omega_1^V$ and $H \in V^{\mathbb{P}}$ which is $(L[a,z],\mathbb{Q})$-generic such that $b \in L[a,z][H]$. By $\mathbf{\Pi}^1_2$-absoluteness we have $L[a,z][H] \vDash \varphi(a)$. Hence there exists $q \in H$ such that $q \Vdash^{L[a,z]}_{\mathbb{Q}} \varphi(a)$.
Since $|\mathbb{Q}| < \omega_1^V$ and $\{a,z\}^\sharp$ exists, we can find an $(L[a,z],\mathbb{Q})$-generic filter H' containing q in V. Hence $L[a,z][H'] \vDash \varphi(a)$ and by $\mathbf{\Sigma}^1_3$-upward absoluteness we have $V \vDash \varphi(a)$. □

Thin, Symmetric $\mathbf{\Delta}^1_3$ Relations

Definition

We call $E \subseteq \omega^\omega \times \omega^\omega$ a symmetric $\mathbf{\Delta}^1_3$ relation iff E has a $\mathbf{\Delta}^1_3$ definition and $\forall x, y \in \omega^\omega : (x, y) \in E \Leftrightarrow (y, x) \in E$.
We call E thin iff there exists no perfect set of pairwise E-incompatible reals.

Theorem (Sch.-Sch.-Sch.)

Let E be a symmetric, (sufficiently) absolute $\mathbf{\Delta}^1_3$ relation, let $\mathbb{P}$ be a countable support iteration of Sacks forcing and assume that $V \vDash$ 'Every real has a sharp'.
If $V \vDash E$ is thin, then $V^{\mathbb{P}} \vDash \forall x \in \omega^\omega\ \exists y \in \omega^\omega \cap V : (x, y) \in E$.

Thin, Symmetric $\mathbf{\Delta}^1_3$ Relations

We will need several Lemmas:

Lemma (1)

Let E be a thin, symmetric $\mathbf{\Pi}^1_3$ relation, let $\dot{\tau}$ be a $\mathbb{P}$-name for a real and assume that $V \models$ 'Every real has a sharp'.
Then the set $D := \{p \in \mathbb{P} \colon (p, p) \Vdash_{\mathbb{P}\times\mathbb{P}} \dot{\tau}^{\dot{G}_1} E \dot{\tau}^{\dot{G}_2}\}$ is dense in $\mathbb{P}$.

Lemma (2)

Let θ be large enough and let $N \prec H(\theta)$ be a countable, elementary submodel with $\mathbb{P} \in N$. Furthermore, let $g \in V$ be an $(N, \mathbb{P})$-generic filter. Then for every $p \in \mathbb{P} \cap N$ there exists $q \leq_{\mathbb{P}} p$ such that $q \Vdash_{\mathbb{P}} g \times (\dot{G} \cap N)$ is $(N, \mathbb{P} \times \mathbb{P})$-generic.

and

Lemma (3)

Assume that ω_1 is inaccessible to the reals. Then $\mathbb{P} \times \mathbb{P}$ is captured.

Thin, Symmetric $\mathbf{\Delta}^1_3$ Relations

Proof (of the Thm.)

Assume towards a contradiction that there exists a condition $p \in \mathbb{P}$ and a $\mathbb{P}$-name for a real $\dot{\tau}$ such that for every $x \in \omega^\omega \cap V$ we have $p \Vdash_{\mathbb{P}} \neg x E \dot{\tau}$.
By Lemma (1) we can assume w.l.o.g. that $(p, p) \Vdash_{\mathbb{P}\times\mathbb{P}} \dot{\tau}^{\dot{G}_1} E \dot{\tau}^{\dot{G}_2}$.
Let θ be large enough and let $N \prec H(\theta)$ be a countable, elementary submodel with $p, \mathbb{P}, \dot{\tau} \in N$. Let $\mathrm{mos}\colon N \to \bar{N}$ denote the Mostowski collapse. Working in V we can now pick an $(N, \mathbb{P})$-generic filter g with $p \in g$. By Lemma (2) we can find $q \leq_{\mathbb{P}} p$ such that $q \Vdash_{\mathbb{P}} g \times (\dot{G} \cap N)$ is $(N, \mathbb{P} \times \mathbb{P})$-generic.
Since $\mathbb{P} \times \mathbb{P}$ is captured by Lemma (3), we can deduce that $q \Vdash_{\mathbb{P}} \bar{N}[\mathrm{mos}[g \times (\dot{G} \cap N)]]$ is closed under sharps. Hence we can deduce that $q \Vdash_{\mathbb{P}} \bar{N}[\mathrm{mos}[g \times (\dot{G} \cap N)]] \prec_{\mathbf{\Sigma}^1_2} V[\dot{G}]$. Since by (1) we have $q \Vdash_{\mathbb{P}} \bar{N}[\mathrm{mos}[g \times (\dot{G} \cap N)]] \models \dot{\tau}^g E \dot{\tau}^{\dot{G}}$, $\mathbf{\Sigma}^1_3$-upward absoluteness implies that $q \Vdash_{\mathbb{P}} \dot{\tau}^g E \dot{\tau}^{\dot{G}}$.
This, however, leads to a contradiction, since $\dot{\tau}^g \in \omega^\omega \cap V$. □

Further Questions

Question

Let $\mathbb{P} = \langle \mathbb{P}_\alpha, \dot{P}_\beta \colon \alpha \leq \kappa, \beta < \kappa \rangle$ be a countable support iteration such that for every $\alpha < \kappa$ we have $\Vdash_{\mathbb{P}_\alpha} \dot{P}_\alpha$ is proper $\wedge$ $\dot{P}_\alpha$ is captured. Does then follow that $\mathbb{P}$ is captured?

Question

Let $\mathbb{P}$ be a countable support iteration of Miller forcing. Assuming ω_1 is inaccessible to the reals, is $\mathbb{P} \times \mathbb{P}$ captured?

Question

Let E be a thin, symmetric, (sufficiently) absolute $\mathbf{\Delta}^1_3$ relation, let $\mathbb{P}$ be either Laver or Mathias forcing and assume that $V \vDash$ 'Every real has a sharp'. Can we again show that $V^{\mathbb{P}} \vDash \forall x \in \omega^\omega \ \exists y \in \omega^\omega \cap V \colon (x, y) \in E$?

Thank you for listening!!!