Independent families and singular cardinals

Diana Carolina Montoya

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University of Vienna
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Section 1

Independent families at uncountable cardinals

Basic definitions

Definition

Assume that κ is a regular cardinal and χ is an infinite cardinal. Let $\mathcal{A}$ be a family of subsets of χ such that $|\mathcal{A}| \geq \kappa$:

- *We denote by* $\mathrm{BF}_\kappa(\mathcal{A})$ *the family of partial functions* $\{h : \mathcal{A} \to 2 : |\mathrm{dom}(h)| < \kappa\}$ *and call it the family of bounded functions on* $\mathcal{A}$.
- *Given* $h \in \mathrm{BF}_\kappa(\mathcal{A})$, *we define*

$$\mathcal{A}^h = \bigcap\{A^{h(A)} : A \in \mathcal{A} \cap \mathrm{dom}(h)\},$$

  where $A^{h(A)} = A$ *if* $h(A) = 0$ *and* $A^{h(A)} = \chi \setminus A$ *otherwise. We call* $\mathcal{A}^h$ *the Boolean combination of* $\mathcal{A}$ *associated to* h *and we refer to* $\{\mathcal{A}^h : h \in \mathrm{BF}_\kappa(\mathcal{A})\}$ *as the family of generalized boolean combinations of the family* $\mathcal{A}$.

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Independent families

Definition

Let κ be a regular cardinal. A family $\mathcal{A} \subseteq \mathcal{P}(\chi)$ such that $|\mathcal{A}| \geq \kappa$ is called κ-independent if for for every $h \in \mathsf{BF}_\kappa(\mathcal{A})$, the set $\mathcal{A}^h$ has size χ.

A κ-independent family $\mathcal{A}$ is said to be maximal κ-independent if it is not properly contained in another κ-independent family. We call the cardinal κ the degree of independence of the family $\mathcal{A}$.

The issue with existence

- Analogously to the classical case ($\chi = \kappa = \omega$) it is possible to construct κ-independent families of size 2^κ (under some assumptions on κ).
- However, it is not possible to use Zorn's lemma to prove the existence of maximal κ-independent families, if κ is uncountable.

The following result of Kunen provides necessary conditions for the existence of maximal κ-independent families in the general context when κ is a regular uncountable cardinal.

Kunen's Theorem

Theorem (See Theorem 1 in [Kun83])

Suppose that κ is regular and uncountable and χ is any infinite cardinal. Also assume that there is a maximal κ-independent family $\mathcal{A} \subseteq \mathcal{P}(\chi)$, with $|\mathcal{A}| \geq \kappa$. Then:

1. $2^{<\kappa} = \kappa$ *and,*
2. *there is a* Γ *with* $\sup\{(2^{\alpha})^{+} : \alpha < \kappa\} \leq \Gamma \leq \min\{\chi, 2^{\kappa}\}$ *such that, there is a non-trivial* κ^{+}*-saturated* Γ*-complete ideal over* Γ*.*

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Saturated Ideals

Definition

Let κ be a cardinal. An ideal $\mathcal{J}$ of subsets of κ is said to be γ-saturated if for any $\{X_\alpha : \alpha < \gamma\} \subseteq \mathcal{J}^+$, there are $\alpha_1, \alpha_2 < \gamma$ such that $X_{\alpha_1} \cap X_{\alpha_2} \in \mathcal{J}^+$. Here $\mathcal{J}^+ = \mathcal{P}(\kappa) \setminus \mathcal{J}$.

For a given ideal $\mathcal{J} \subseteq \mathcal{P}(\kappa)$ being γ-saturated is equivalent to the Boolean algebra $\mathcal{P}(\kappa)/\mathcal{J}$ having the γ-cc. Let $\text{Sat}(\theta, \gamma, \mathcal{J})$ abbreviate the statement "$\mathcal{J}$ is a θ-complete, γ-saturated ideal" and $\text{Sat}(\theta, \gamma)$ the statement: "There is an ideal $\mathcal{J}$ that is θ-complete and γ-saturated ideal".

Notice that the property Sat$(\theta, \gamma, \mathcal{J})$ gets weaker when γ increases, i.e. if $\gamma < \gamma'$ then Sat$(\theta, \gamma, \mathcal{J}) \rightarrow$ Sat$(\theta, \gamma', \mathcal{J})$. Also Sat$(\theta, \omega)$ is equivalent to κ being measurable.

We will use the following result:

Theorem (Prikry, Solovay and Kakuda. See Theorem 17.1 in [Kan03])

Suppose that $\mathcal{J}$ is a δ-saturated ideal over κ, where $\delta \leq \kappa^+$ is regular and $\mathbb{P}$ is a partial order with the ν-cc where $\nu < \kappa$ and $\nu \leq \delta$. Then:

$$\Vdash_{\mathbb{P}} \check{\mathcal{J}} \textit{ generated a } \delta - \textit{saturated ideal over } \kappa$$

Coming back to Kunen's result

▸ Kunen's Theorem

- Since $\Gamma \geq \kappa$, then the ideal given by the theorem must be Γ^+-saturated, which yields to an inner model with a measurable cardinal.
- If κ is not strongly inaccessible then $\Gamma \geq \kappa^+$, which implies by Ulam that κ is weakly inaccessible and Solovay that is is also weakly Mahlo.
- If κ is strongly inaccessible, it is consistent that $\kappa = \Gamma = \chi$.

A comment on countable independence degree and the regular case

If we assume $\kappa = \omega$ the existence of maximal κ-independent families at a cardinal χ is a straightforward consequence of Zorn's lemma. The following is a result of Fischer and myself regarding these families.

Theorem (See [FM20])

Let χ be a measurable cardinal and let $2^\chi = \chi^+$. Then there is a maximal ω-independent family of subsets of χ, which remains maximal after the χ-support product of δ-many copies of χ-Sacks forcing.

Also, Eskew and Fischer have studied the concept of independence for regular cardinals. In [EF21] they prove in particular that if $\mathfrak{i}(\kappa)$ is the minimum size of a maximal κ-independent family of subsets of κ. Then, it is consistent that $\kappa^+ < \mathfrak{i}(\kappa) < 2^\kappa$.

They also studied the spectrum of maximal κ-independent families at χ and gave a wide set of results involving it.

Section 2

Kunen's proof

We review a few details of the proof of ▸ Kunen's Theorem which will be relevant for the results to come. Suppose that κ is a regular cardinal and let $\mathcal{A}$ be a κ-*maximal* independent family of subsets of χ.

Define the map

$$\begin{aligned} \varphi\colon \mathrm{Fn}_{<\kappa}(\mathcal{A}, 2) &\to \mathcal{P}(\chi) \\ p &\mapsto \mathcal{A}^p. \end{aligned}$$

where $\mathrm{Fn}_{\kappa}(\mathcal{A}, 2)$ is the classical poset of partial functions $p : \mathcal{A} \to 2$ with $|\mathrm{dom}(p)| < \kappa$.

The map φ

- φ is an isomorphism from $\mathrm{Fn}_\kappa(\mathcal{A}, 2)$ into $[\chi]^\chi$.
- $p \leq q$ implies $\varphi(p) \subseteq \varphi(q)$.
- Two conditions p, q are compatible in $\mathbb{P} = \mathrm{Fn}_\kappa(\mathcal{A}, 2)$ if and only if $\varphi(p) \cap \varphi(q) \neq \emptyset$.
- The family $\mathcal{A}$ is maximal if and only if for all $X \subseteq \chi$ there is a $p \in \mathbb{P}$ such that $\varphi(p) \subseteq^* X$ or $\varphi(p) \subseteq^* \chi \setminus X$.
- We can even assume that $\mathcal{A}$ is maximal in a stronger sense that we call *densely maximal*, meaning that for all $X \subseteq \chi$ and all $p \in \mathbb{P}$, there is a $q \leq p$ such that $\varphi(q) \subseteq^* X$ or $\varphi(q) \subseteq^* \chi \setminus X$.

One associated ideal

Define the following ideal

$$\mathcal{J}_{\mathcal{A}} := \{X \subseteq \chi : \forall p \in \mathbb{P}\,(\varphi(p) \not\subseteq^* X)\}.$$

To finish the proof of the Theorem, Kunen proved that the ideal $\mathcal{J}_{\mathcal{A}}$ is $(2^{\alpha})^{+}$-complete for all $\alpha < \kappa$, that it is $(2^{<\kappa})^{+}$-saturated and that $2^{<\kappa} = \kappa$ and so $\mathcal{J}_{\mathcal{A}}$ is in fact, κ^{+}-saturated. Hence if Γ is the minimum cardinal such that $\mathcal{J}_{\mathcal{A}}$ is not Γ-complete, one gets the desired result.

Sufficient conditions

Lemma

Suppose κ is regular, $2^{<\kappa} = \kappa$, $\kappa \leq \chi$ and $\mathcal{J}$ is a κ^+-saturated χ-complete ideal over χ such that $\mathcal{B}(\mathsf{Fn}_\kappa(2^\chi, 2))$ isomorphic to $\mathcal{P}(\chi)/\mathcal{J}$.
Then, there is a maximal κ-independent family of subsets of χ.

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A consistency result

Theorem (Kunen)

If there is a measurable cardinal, then there is a maximal σ-independent family $\mathcal{A} \subseteq \mathcal{P}(2^{\omega_1})$.

The proof

- Start with a measurable cardinal κ in a ground model V where CH holds.
- Let $\mathcal{U}$ be a normal measure witnessing the measurability of κ.
- We shall construct a model in which CH still holds and if $\kappa = 2^{\aleph_1}$, there is an ω_2-saturated, κ-complete ideal $\mathcal{J}$ over κ such that the Boolean algebras $\mathcal{P}(\kappa)/\mathcal{J}$ and $\mathcal{B}(\mathrm{Fn}_{\omega_1}(2^\kappa, 2))$ are isomorphic.

Sufficient conditions

- Let $\mathbb{P}$ be $\mathsf{Fn}_{\omega_1}(\kappa, 2)$ and let G to be a $\mathbb{P}$-generic filter over V. In $V[G]$, $\kappa = 2^{\aleph_1}$ and we can define the following collection of subsets of κ:

$$\mathcal{J} = \{X \subseteq \kappa : \exists Y \in \mathcal{U}(X \cap Y = \emptyset)\}$$

- $\mathcal{J}$ is, in turn a κ-complete ω_2-saturated ideal because $\mathbb{P}$ has the ω_2-cc and so $\mathcal{J}$ is ω_2-saturated and κ-complete in $V[G]$.

The rest of the argument aims to construct an isomorphism between the Boolean algebras $\mathcal{P}(\kappa)/\mathcal{I}$ and $\mathcal{B}(\mathrm{Fn}_{\omega_1}(2^\kappa, 2))$ in $V[G]$.

- Let $j : V \to M = \mathrm{Ult}(V, \mathcal{U})$ be the ultrapower embedding associated to $\mathcal{U}$, i.e. j is elementary, $\mathrm{crit}(j) = \kappa$.
- Let $\kappa^* = j(\kappa) > \kappa$, then $2^\kappa < \kappa^* < (2^\kappa)^+$ and the posets $\mathrm{Fn}_{\omega_1}(2^\kappa, 2)$ and $\mathrm{Fn}_{\omega_1}(\kappa^* \backslash \kappa, 2)$ are isomorphic.

The isomorphism

Let's define the isomorphism $\Gamma : \mathcal{P}(\kappa)/\mathcal{J} \to \mathcal{B}(\mathsf{Fn}_{\omega_1}(\kappa^*\backslash\kappa, 2))$ in $V[G]$ as follows: Given $[X] \in (\mathcal{P}(\kappa)/\mathcal{J})^{V[G]}$, and let $\dot{X}$ be a $\mathbb{P}$-name for the set X. We define the function as follows:

$$\Gamma([X]) := \bigvee\{q \in \mathsf{Fn}_{\omega_1}(\kappa^*\backslash\kappa, 2) : \exists p \in G(p \cup q \Vdash \check{\kappa} \in j(\dot{X}))\}.$$

- Recall that $j(\mathbb{P}) = j(\mathrm{Fn}_{\omega_1}(\kappa, 2)) = \mathrm{Fn}_{\omega_1}(\kappa^*, 2) \simeq \mathbb{P} \times \mathbb{Q}$, where $\mathbb{Q} = \mathrm{Fn}_{\omega_1}(\kappa^* \setminus \kappa, 2)$. Also, every element of the poset $\mathbb{Q}$ is represented in $\mathrm{Ult}(V, \mathcal{U})$ by a sequence $(q_\alpha : \alpha < \kappa)$ such that $q_\alpha \in \mathbb{Q}$ for all $\alpha < \kappa$.
- Thus, if H is $\mathbb{Q}$-generic over $V[G]$, then $G \times H$ is $j(\mathbb{P})$-generic over V and we can define a map j to $j^* : V[G] \to M[G \times H]$ as $j^*(X) = (j(\dot{X}))^{G \times H}$ in $V[G \times H]$. So, we can ask for a given set $Y \in V[G]$ whether or not $\check{\kappa} \in (j(\dot{Y}))^{G \times H}$.

Two more consistency results

Corollary

Assume κ is strongly compact in V. Then in $V[G]$, where G is $\mathbb{P}$-generic (for $\mathbb{P} = \mathrm{Fn}_{\omega_1}(\kappa, 2)$ like in the theorem above) for every cardinal $\chi \geq \kappa$ such that $\mathrm{cf}(\chi) \geq \kappa$ there is a maximal σ-independent family of subsets of χ.

Theorem

Let δ be a regular cardinal such that $2^{<\delta} = \delta$ and κ be a measurable cardinal above it. Then there is a maximal δ-independent family $\mathcal{A} \subseteq \mathcal{P}(2^\delta)$.

Section 3

The singular case

Framework

Now, we want to study the concept of independence in the case when λ is a singular cardinal of cofinality $\kappa < \lambda$.

Look at the definition of ▸ Independence and notice, there is no a priori restriction about lifting it to the context of a singular.
Note that if $\mathcal{A}$ is λ-independent, then it is λ'-independent for all $\lambda' < \lambda$; in particular cf$(\lambda) = \kappa$-independent. The other direction does not hold:

Hausdorff's example at $\aleph_\omega$

Let

$$\mathcal{C} = \{(a, A) : a \in [\lambda]^{<\omega}, A \subseteq \mathcal{P}(a)\}$$

and note $|\mathcal{C}| = \aleph_\omega^{<\omega} = \aleph_\omega$.

For $X \subseteq \lambda$ define

$$\mathcal{Y}_X = \{(a, A) \in \mathcal{C} : X \cap a \in A\}.$$

Then, $\mathcal{A} = \{\mathcal{Y}_X : X \subseteq \lambda\} \subseteq \mathcal{P}(\mathcal{C}) \simeq \mathcal{P}(\aleph_\omega)$ is ω-independent (or σ-independent).

Given $X_0, X_1, \ldots X_i$ and $Z_0, Z_1, \ldots Z_j$ for $i, j < \omega$, if $a \in [\lambda]^{<\omega}$ is such that $X_l \cap a \neq X_{l'} \cap a \neq Z_n \cap a \neq Z_{n'} \cap a$ for all $l, l' \leq i$ and $n, n' \leq j$. Then $a \in \bigcap_{l \leq i} \mathcal{Y}_{X_l} \cap \bigcap_{l \leq j} \lambda \backslash \mathcal{Y}_{Z_j}$.

- Notice that $\mathcal{A}$ is not ω_1-independent: If $X_0 \subseteq X_1 \subseteq \ldots X_n \subseteq \ldots$ is cofinal in λ. Take $(a, A) \in \bigcap_{i \text{ even}} \mathcal{Y}_{X_i} \cap \bigcap_{i \text{ odd}} \lambda \backslash \mathcal{Y}_{X_i}$. Since the sequence of the X_n's is cofinal there is a $n_a \in \omega$ (we can take it minimal) such that $a \subseteq X_{n_a}$, but then for all $i \geq n_a$, $a \cap X_i = a$ which is a contradiction.

More simple properties

The former is a general behavior:

Proposition

Let λ be a singular cardinal of cofinality $\kappa < \lambda$. Suppose that $\mathcal{A}$ is a κ-independent family of subsets of λ, then $\mathcal{A}$ is not κ^+-independent.

Proposition

Suppose λ is a strong limit singular cardinal with $\mathrm{cf}(\lambda) = \kappa$. Then there is a κ-independent family of subsets of λ of size 2^λ.

Maximality

Now we turn into maximality and the issue of existence of maximal independent families at singular cardinals. From now on, we assume that λ is a singular cardinal of cofinality $\kappa < \lambda$.

First we establish that a κ-independent family $\mathcal{A} \subseteq [\lambda]^\lambda$ is **maximal** if for all $X \in [\lambda]^\lambda$ there is a bounded function $\mathrm{BF}_\kappa(\mathcal{A})$ such that either $\mathcal{A}^h \setminus X$ or $\mathcal{A}^h \cap X$ is bounded in λ (i.e. of size $< \lambda$).

Cases

- Let's consider the case where λ is singular of countable cofinality. In this case existence of a maximal ω-independent family (or just *independent*) of subsets of λ can be proven using Zorn's lemma.
- In the case of λ singular of cofinality $\kappa > \omega$ we have the following: if there exists $\mathcal{A} \subseteq [\lambda]^\lambda$ a maximal κ-independent family, then Kunen's Theorem implies that $2^{<\kappa} = \kappa$ and that there is an ordinal Γ with $\sup\{(2^\alpha)^+ : \alpha < \lambda\} \leq \Gamma \leq \min\{\lambda, 2^\kappa\}$ such that, there is a non-trivial κ^+-saturated Γ-complete ideal over Γ.

Our results

The next result guarantees the existence of a maximal κ-independent family at a singular cardinal λ of cofinality κ, when we assume the existence of maximal κ-independent families at cardinals $(\lambda_\alpha : \alpha < \kappa)$ converging to λ.

Lemma

Assume that λ is a singular cardinal of cofinality κ which is a limit of the sequence of cardinals $(\lambda_\alpha : \alpha < \kappa)$ of regular cardinals such that, for each $\alpha < \kappa$, there is a maximal δ-independent family $\mathcal{A}_\alpha \subseteq [\lambda_\alpha]^{\lambda_\alpha}$ and $\delta \leq \kappa < \lambda_0$ is regular such that there is a maximal δ-independent family of subsets of κ. Then, there is a maximal δ-independent family $\mathcal{B} \subseteq [\lambda]^\lambda$.

Lemma (An improvement of the lemma above)

Assume that λ is a singular cardinal of cofinality κ which is a limit of the sequence of cardinals $(\lambda_\alpha : \alpha < \kappa)$. Let also $(\delta_\alpha : \alpha < \kappa)$ be a sequence of regular cardinals with limit κ. Suppose also that for each $\alpha < \kappa$, there is a maximal δ_α-independent family $\mathcal{A}_\alpha \subseteq [\lambda_\alpha]^{\lambda_\alpha}$ and $\kappa < \delta_0$ is regular such that there is a maximal κ-independent family of subsets of κ. Then, there is a maximal κ-independent family $\mathcal{B} \subseteq [\lambda]^\lambda$.

Theorem

Start with a ground model V in which GCH *holds. Suppose that λ is a singular cardinal of cofinality κ which is a limit a sequence of cardinals $(\lambda_\alpha : \alpha < \kappa)$. Let also $(\delta_\alpha : \alpha < \kappa)$ be a sequence of regular cardinals converging to κ so that $\alpha \leq \delta_\alpha^{<\delta_\alpha} = \delta_\alpha$ and κ_α is δ_α-supercompact for all $\alpha < \kappa$. Then there is a generic extension of a universe $V \models$* GCH *such that:*

$$V^{\mathbb{P}} \models \text{ There is a maximal } \kappa\text{-independent family of subsets of } \lambda$$

A refinement

Theorem

Assume that λ is a singular cardinal of cofinality κ which is a strong limit of the sequence of cardinals $(\lambda_\alpha : \alpha < \kappa)$. Let also $(\delta_\alpha : \alpha < \kappa)$ be a sequence of regular cardinals with limit κ. Suppose also that for each $\alpha < \kappa$, there is a maximal δ_α-independent family $\mathcal{A}_\alpha \subseteq [\lambda_\alpha]^{\lambda_\alpha}$ of size ρ_α and $\kappa < \delta_0$ is regular such that there is a maximal κ-independent family of subsets of κ. Put also $\chi_{\bar{\lambda}} = \mathrm{tcf}(\Pi_{i<\kappa}\lambda_i, <^*)$ and $\chi_{\bar{\rho}} = \mathrm{tcf}(\Pi_{i<\kappa}\rho_i, <^*)$.

Then, there is a maximal κ-independent family $\mathcal{B} \subseteq [\lambda]^\lambda$ of cardinality $\chi_{\bar{\lambda}} \cdot \chi_{\bar{\rho}}$.

Sizes of independent families (work in progress)

Let λ be a singular cardinal of cofinality $\kappa < \lambda$, let's define:

$$\mathfrak{i}(\lambda) = \{|\mathcal{A}| \colon \mathcal{A} \subseteq [\lambda]^\lambda \text{ such that } \mathcal{A} \text{ is maximal } \kappa\text{-independent}\}$$

- Eskew-Fischer's results.
- Shelah's result on $\mathfrak{d}(\lambda)$.
- The main open question.

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