Asymptotic differential algebra and logarithmic transseries

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du Bois - Reymond (1871–1882) “calculation of infinities”
Borel (1899)
Hahn (1907)
Hausdorff (1909, 1914)
Hardy (1911–1928) “Orders of infinity”, log-exp functions
Rosenlicht (1972–1995)
Ecalle (1975–1993) Dulac Problem
Bourbaki (1976) Functions of a real variable, Hardy Field appendix
vander Hoeven (1977–present)
Aschenbrenner (2000–present)
vander Dries (2000–present)

Asymptotic Differential Algebra and Model Theory of Transseries
Discrete

Geometric series \((r > 0)\)

\[ \sum r^n \begin{cases} \text{converges if } r < 1 \\ \text{diverges if } r \geq 1 \end{cases} \]

Continuous

\[ \int r^x \, dx \begin{cases} \text{converges if } r < 1 \\ \text{diverges if } r \geq 1 \end{cases} \]

This plus comparison test can determine any function “visible at the exponential level” but cannot determine functions “visible at the polynomial level” since \( (1-\varepsilon)^n < \frac{1}{n^2} < 1 \) \((0 > \varepsilon)\) as \( n \to +\infty \)

But we can convert polynomial level functions to the exponential level with

Cauchy Condensation Test \( (a_n) \) positive decreasing

\[ \sum a_n \text{ converges } \iff \sum 2^n a_{2^n} \text{ converges} \]

E.g. \( \sum \frac{1}{n^3} \text{ converges } \iff \sum \frac{2^n}{(2^n)^{1+3}} \text{ converges} \]

\[ \iff \sum \left( \frac{1}{2^3} \right)^n \text{ converges} \iff \sum > 0 \]
Can repeat this to get the "logarithmic criterion of order 1"

$$\sum \frac{1}{\ln(n)^{1+\varepsilon}} \text{ converges iff } \varepsilon > 0$$

$$\int_1^\infty \frac{dx}{x (\ln x)^{1+\varepsilon}} \text{ converges iff } \varepsilon > 0$$

and more generally:

$$\sum \frac{\ln^{l+1}(n)}{\ln(n)!} \text{ converges iff } \varepsilon > 0$$

$$\int_1^\infty \frac{dx}{\ln x \cdot \ln^{l+1}(n)} \text{ converges iff } \varepsilon > 0$$

where \( n_0 = x, \ l_1 = \ln x, \ l_{k+1} = \ln(l_k) \).

To summarize:

$$\int_0^\infty \text{ diverges } \quad \int_0^\infty \text{ converges}$$

$$\int_0^1 \frac{dx}{x} \quad \int_0^1 \frac{dx}{x^{1+\varepsilon}}$$

$$\int_0^1 \frac{dx}{x \ln x} \quad \int_0^1 \frac{dx}{x (\ln x)^{1+\varepsilon}}$$

\( \varepsilon > 0 \).
Hahn fields (or generalized power series): a construction

- Let a field $C$ and an ordered (multiplicative) abelian group of “monomials” $\mathbb{M} = (\mathbb{M}; \cdot, \prec)$ be given.

- A set $\mathcal{S} \subseteq \mathbb{M}$ is **well-based** if there is no strictly increasing sequence $m_0 \prec m_1 \prec m_2 \prec \cdots$ in $\mathcal{S}$.

- Given a function $f : \mathbb{M} \to C$, written as a formal series $\sum_{m \in \mathbb{M}} f(m) m$ with $f(m) \equiv f(m)$, the **support** of $f$ is $\text{supp } f := \{m \in \mathbb{M} : f(m) \neq 0\}$.

- The **Hahn field** $\mathbb{C}[[\mathbb{M}]] := \{f : \mathbb{M} \to C : \text{supp } f \text{ is well-based}\}$ is a valued field with pointwise addition and “series multiplication” (Neumann’s lemma) with residue field $\mathbb{C}$. Value group is an additive copy of $\mathbb{M}$ with reverse ordering.

- Example: $\mathbb{C}[[t^\mathbb{Z}]]$, where $t := x^{-1}$, is the same as usual field of Laurent series $\mathbb{C}((x))$. 
The (Ordered) Valued Field $\mathbb{T}_\text{log}$

**Definition (The valued field $\mathbb{T}_\text{log}$ of logarithmic transseries)**

$$\mathbb{T}_\text{log} := \bigcup_n \mathbb{R}[[\mathcal{L}_n]]$$

union of spherically complete Hahn fields

where $\mathcal{L}_n$ is the ordered group of logarithmic transmonomials:

$$\mathcal{L}_n := \ell_0^R \cdots \ell_n^R = \{\ell_0^r \cdots \ell_n^r : r_i \in \mathbb{R}\}, \quad \ell_0 = x, \ell_{m+1} = \log \ell_m$$

ordered such that $\ell_i > \ell_i^r > 1$ for all $r \in \mathbb{R}^+, i = 0, \ldots, n-1$.

Typical elements of $\mathbb{T}_\text{log}$ look like:

- $-2x^3 \log x + \sqrt{x} + 2 + \frac{1}{\log \log x} + \frac{1}{(\log \log x)^2} + \cdots$

- $\frac{1}{\log \log x} + \frac{1}{(\log \log x)^2} + \cdots + \frac{1}{\log x} + \frac{1}{(\log x)^2} + \cdots + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \cdots$

Note: $\mathbb{T}_\text{log}$ is a real closed field and thus has a definable ordering.

Also: Residue field is $\mathbb{R}$ and value group $\Gamma_\text{log}$ is additive copy of $\bigcup_n \mathcal{L}_n$ with reverse ordering.

\[
\frac{1}{x} + \frac{1}{x^2} + \frac{1}{\ell_0 \cdots \ell_2} + \frac{1}{\ell_0 \cdots \ell_3} + \cdots \in \mathbb{T}_\text{log}
\]
The derivation on $\mathbb{T}_{\log}$

$\mathbb{T}_{\log}$ comes equipped with the usual termwise derivative and logarithmic derivative:

$$f \mapsto f'$$

$$f \mapsto f^\dagger := f'/f, \quad (f \neq 0)$$

subject to the usual rules: $\ell_0' = 1$, $\ell_1' = \ell_0^{-1}$, etc.

For example:

- $(x^3 \log x + \sqrt{x} + 2 + \cdots)' = 3x^2 \log x + x^2 + \frac{1}{2x^{1/2}} + \cdots$
- $(\ell_n^\dagger = \frac{1}{\ell_0 \ell_1 \cdots \ell_n}$
- $(\frac{1}{\log \log x} + \frac{1}{(\log \log x)^2} + \cdots)' = -\frac{1}{x \log x (\log \log x)^2} - \frac{2}{x \log x (\log \log x)^3} + \cdots$
- $(\ell_0^{r_0} \cdots \ell_n^{r_n})^\dagger = r_0 \ell_0^{-1} + r_1 \ell_0^{-1} \ell_1^{-1} + \cdots + r_n \ell_0^{-1} \cdots \ell_n^{-1}$

This derivative makes $\mathbb{T}_{\log}$ into a differential field with field of constants $\mathbb{R}$. 
**Definition**

$K$ an ordered valued differential field. We call $K$ an **$H$-field** if

- **H1** for all $f \in K$, if $f > C$, then $f' > 0$;
- **H2** $\mathcal{O} = C + \mathfrak{o}$ where $\mathcal{O} = \{ g \in K : |g| \leq c \text{ for some } c \in C \}$ is the (convex) valuation ring of $K$ and $\mathfrak{o}$ is the maximal ideal of $\mathcal{O}$.

**Example**

$\mathbb{T}_{\log}$ is an $H$-field, also any Hardy field containing $\mathbb{R}$ is an $H$-field.

**Example**

$\mathbb{T}$, the differential field of **logarithmic-exponential transseries** is naturally an $H$-field, and contains $\mathbb{T}_{\log}$. It is closed under exp. Typical element:

$$-3e^x + e^{\frac{e^x}{\log x}} + e^{\frac{e^x}{\log^2 x}} + e^{\frac{e^x}{\log^3 x}} + \cdots - x^{11} + 7\frac{\pi}{x} + \frac{1}{x \log x} + \cdots + e^{-x} + 2e^{-x^2} + \cdots$$
The asymptotic couple \((\Gamma, \psi)\) of an \(H\)-field \(K\)

**Fact**

For \(f \in K^\times\) such that \(\nu(f) \neq 0\), the values \(\nu(f')\) and \(\nu(f^\dagger)\) depend only on \(\nu(f)\).

\[
\begin{array}{ccc}
K & \xrightarrow{'} & K \\
\downarrow \nu & & \downarrow \nu \\
\Gamma & \xrightarrow{\cdots} & \Gamma \\
\end{array}
\quad
\begin{array}{ccc}
K & \xrightarrow{\dagger} & K \\
\downarrow \nu & & \downarrow \psi \\
\Gamma & \xrightarrow{\cdots} & \Gamma \\
\end{array}
\]

\((\Gamma\ is\ the\ value\ group\ of\ K)\)

**Definition (Rosenlicht)**

The pair \((\Gamma, \psi)\) is the asymptotic couple of \(K\).

**Theorem (G)**

\(\text{Th}(\Gamma_{\log}, \psi)\), the asymptotic couple of \(\mathbb{T}_{\log}\), has QE in a natural language, is model complete, has NIP, and is distal (with Elliot Kaplan, 2018).
$H$-fields: two technical properties

Both $\mathcal{T}$ and $\mathcal{T}_{\log}$ enjoy two additional (first-order) properties:

- **ω-free**: this is a very strong and robust property which prevents certain deviant behavior
  \[ \forall f \neq 0 \exists g \neq 1[g' \asymp f] \quad \& \quad \forall f \exists g > 1 [f + 2g^{\uparrow \uparrow'} + 2(g^{\uparrow \uparrow})^2 \asymp (g^{\uparrow})^2] \]

- **newtonian**: this is a variant of “differential-henselian”; it essentially means that you can simulate being differential henselian arbitrarily well by sufficient coarsenings and compositional conjugations ($\partial \mapsto \phi \partial$).

$\mathcal{T}_{\log}$ satisfies both of these properties because it has integration and is a union of spherically complete $H$-fields, each with a smallest “comparability class”:

\[ \mathcal{T}_{\log} := \bigcup_n \mathbb{R}[[\ell_0^\mathbb{R} \cdots \ell_n^\mathbb{R}]] \]
Another nice property:

**Definition**

We call a real closed $H$-field $K$ **Liouville closed** if

$$K' = K \quad \text{and} \quad (K^x)^\dagger = K$$

$\mathbb{T}$ is Liouville closed, however...

$\mathbb{T}_{\log}$ is NOT Liouville closed:

$$(\mathbb{T}_{\log})' = \mathbb{T}_{\log} \quad \text{but} \quad (\mathbb{T}_{\log}^x)^\dagger \neq \mathbb{T}_{\log}$$

E.g., an element $f$ such that $f^\dagger = 1$ would have to behave like $e^x$.
The field $\mathbb{T}$: a success story

Let $\mathcal{L} = \{0, 1, +, -, \cdot, \partial, \leq, \preceq\}$

The following result is the starting point for the model theory of $\mathbb{T}_{\log}$:

**Theorem (Aschenbrenner, van den Dries, van der Hoeven, 2015)**

$\mathbb{T}$ is model complete as an $\mathcal{L}$-structure. Furthermore, $\text{Th}_{\mathcal{L}}(\mathbb{T})$ is axiomatized by:

- real closed, $\omega$-free, newtonian, $H$-field such that $\forall \varepsilon < 1, \partial(\varepsilon) < 1$;
- Liouville closed
  - $K' = K$
  - $(K^\times)^\dagger = K$

Recall: a structure $M$ is model complete if every definable subset of $M^n$ is existentially definable (for every $n$). A starting point for model completeness of $\mathbb{T}_{\log}$ is to try to make both $(\mathbb{T}_{\log}^\times)^\dagger$ and its complement existentially definable.
Investigating \((\mathbb{T}_\log^\times)\)

\[f \in (\mathbb{T}_\log^\times) \iff \text{there exists } g \in \mathbb{T}_\log^\times \text{ such that } g^\dagger = f\]

Given \(f \in \mathbb{T}_\log^\times\), we can write it uniquely as

\[f = c \ell_0^{r_0} \cdots \ell_n^{r_n}(1 + \epsilon) \text{ for some infinitesimal } \epsilon < 1 \text{ and some } c \in \mathbb{R}_\times\]

Then we compute the logarithmic derivative:

\[(c \ell_0^{r_0} \cdots \ell_n^{r_n}(1 + \epsilon))^\dagger = r_0 \ell_0^{-1} + r_1 \ell_0^{-1} \ell_1^{-1} + \cdots + r_n \ell_0^{-1} \cdots \ell_n^{-1} + \frac{\epsilon'}{1 + \epsilon}\]

“small”

Note: \(\nu(\ell_0^{-1} \cdots \ell_n^{-1}) \in \Psi := \psi(\Gamma_{\log}^\neq) \text{ and } \nu(\epsilon'/(1 + \epsilon)) > \Psi\).

Fact

\[f \notin (\mathbb{T}_\log^\times) \iff \text{there exists } g \in \mathbb{T}_\log^\times \text{ such that } \nu(f - g^\dagger) \in \Psi \setminus \Psi\]
Introducing LD-\(H\)-fields

From now on all \(H\)-fields will have asymptotic integration \((\Gamma = (\Gamma \neq)^\prime)\).
Let \(K\) be an \(H\)-field and \(LD \subseteq K\).
We call the pair \((K, LD)\) an LD-\(H\)-field if:

**LD1** LD is a \(C_K\)-vector subspace of \(K\);

**LD2** \((K^\times)^\dagger \subseteq LD\);

**LD3** \(I(K) := \{y \in K : y \ll f' \text{ for some } f \in \mathcal{O}\} \subseteq LD\); and

**LD4** \(v(LD) \subseteq \Psi \cup (\Gamma^\geq)^\prime \cup \{\infty\}\).

We say an LD-\(H\)-field \((K, LD)\) is **full** if:

**full** For every \(a \in K \setminus LD\), there is \(b \in LD\) such that \(v(a - b) \in \Psi^\perp \setminus \Psi\),

and we say it is \(\Psi\)-closed if it is full and \(LD = (K^\times)^\dagger\).

**Example**

\((T_{\log}, (T_{\log}^\times)^\dagger)\) and \((T, T)\) are both \(\Psi\)-closed LD-\(H\)-fields.
Let $\mathcal{L}_{LD} := \{0, 1, +, -, \cdot, \partial, \leq, \preceq, LD\}$ where LD is a unary relation symbol.
Let $T_{log}$ be the $\mathcal{L}_{LD}$-theory whose models are precisely the LD-$H$-fields $(K, LD)$ such that:

1. $K$ is real closed, $\omega$-free, and newtonian;
2. $(K, LD)$ is $\Psi$-closed; and
3. $(\Gamma, \psi) \models \text{Th}(\Gamma_{log}, \psi)$, where $(\Gamma, \psi)$ is the asymptotic couple of $K$.

**Conjecture**

The theory $T_{log}$ is model complete.

**Embedding version of conjecture**

Let $(K, LD)$ and $(L, LD_1)$ be models of $T_{log}$ and suppose $(E, LD_0)$ is a full $\omega$-free LD-$H$-subfield of $(K, LD)$ such that $(\mathbb{Q} \Gamma_E, \psi) \models \text{Th}(\Gamma_{log}, \psi)$. Let $i : (E, LD_0) \to (L, LD_1)$ be an embedding of LD-$H$-fields. Assume $(L, LD_1)$ is $|K|^+$-saturated and $(K, LD)$ is $\aleph_0$-saturated. Then $i$ extends to an embedding $(K, LD) \to (L, LD_1)$ of LD-$H$-fields.
Given LD-$H$-fields $(K, LD)$ and $(L, LD^*)$ such that $K \subseteq L$, we say that $(L, LD^*)$ is an extension of $(K, LD)$ (notation $(K, LD) \subseteq (L, LD^*)$) is $LD^* \cap K = LD$.

**Proposition**

Suppose $L$ is an algebraic extension of $K$, $(K, LD)$ is full, and $(\Gamma, \psi) \models \text{Th}(\Gamma_{\text{log}}, \psi)$. Then there is a unique LD-set $LD^* \subseteq L$ such that $(K, LD) \subseteq (L, LD^*)$; equipped with this LD-set, $(L, LD^*)$ also is full. Important case: $L$ is a real closure of $K$. 
Suppose $K \subseteq L$ is an extension of $H$-fields such that $L = K(C_L)$, so $L$ is a constant field extension of $K$.

**Proposition**

Suppose $K$ is henselian, $(\Gamma, \psi) \models \text{Th}(\Gamma_{\log}, \psi)$, and $(K, LD)$ is full. Then there is a unique LD-set $LD^* \subseteq L$ such that $(K, LD) \subseteq (L, LD^*)$; equipped with this LD-set, $(L, LD^*)$ also is full.

Thus adding new constants will never be an issue!
The $\Psi$-closure of an LD-$H$-field

Definition

We say an LD-$H$-field extension $(K^\Psi, LD^\Psi)$ of $(K, LD)$ is a $\Psi$-closure of $(K, LD)$ if $K^\Psi$ is real closed, $(K^\Psi, LD^\Psi)$ is $\Psi$-closed, and for any LD-$H$-field extension $(L, LD^*)$ of $(K, LD)$ such that $L$ is real closed and $(L, LD^*)$ is $\Psi$-closed, there is an embedding $(K^\Psi, LD^\Psi) \rightarrow (L, LD^*)$ of LD-$H$-fields over $(K, LD)$.

Theorem

Suppose $(K, LD)$ is full, is $\lambda$-free, and $(\Gamma, \psi) \models \text{Th}(\Gamma_{\log}, \psi)$. Then $(K, LD)$ has a unique $\Psi$-closure. This $\Psi$-closure will be differentially-algebraic over $K$, does not contain any proper real closed and $\Psi$-closed differential subfields containing $K$, and its asymptotic couple will model $\text{Th}(\Gamma_{\log}, \psi)$.
Newtonization: a reduction to the linear case

Suppose $K$ is $\omega$-free, $(\Gamma, \psi) \models \text{Th}(\Gamma_{\log}, \psi)$ and let $K^{nt}$ be the newtonization of $K$ (a newtonian extension of $K$ with a natural universal property).

What we would like to prove:

Suppose $(K, LD)$ is full. Then $LD^{nt} := LD + I(K^{nt})$ is the unique LD-set on $K^{nt}$ such that $(K, LD) \subseteq (K^{nt}, LD^{nt})$; equipped with this LD-set, $(K^{nt}, LD^{nt})$ also is full.

It suffices(!!!) to prove the linear case:

Conjecture 1 (Linear newtonian conjecture)

There is a linearly newtonian $H$-field $L$ such that $K \subseteq L \subseteq K^{nt}$ and $LD^* := LD + I(L)$ is the unique LD-set on $L$ such that $(K, LD) \subseteq (L, LD^*)$; equipped with this LD-set, $(L, LD^*)$ also is full.

Linearly newtonian is the fragment of newtonian that only involves degree 1 differential polynomials (differential operators).
Conjecture 2 (Immediate differentially-transcendental conjecture)

Suppose \((L, LD^*)\) is an LD-\(H\)-field extension of \((K, LD)\) such that \((K, LD), (L, LD^*) \models T_{\log}\), and suppose there is \(y \in L \setminus K\) such that \(K\langle y\rangle\) is an immediate extension of \(K\) (so \(y\) is necessarily differentially transcendental over \(K\) since \(K\) is asymptotically d-algebraically maximal). Then \(LD_y := LD + I(K\langle y\rangle)\) is the unique LD-set on \(K\langle y\rangle\) such that \((K, LD) \subseteq (K\langle y\rangle, LD_y)\); equipped with this LD-set, \((K\langle y\rangle, LD_y)\) also is full.

Note: there are weaker versions of Conjectures 1 and 2 which will suffice for our purposes (in case as written they are false).
The main “result”

**Theorem (G)**

Assume Conjectures 1 and 2 hold. Then $T_{\text{log}}$ is model complete as an LD-H-field.
Recall from calculus/ODEs:

- The differential equation

\[ Y' - \cos t = 0 \]

has solutions

\[ \{ \sin t + c_0 : c_0 \in \mathbb{R} \} \]

- The differential equation

\[ Y'' - 3Y' + 2Y = 0 \]

has solutions

\[ \{ c_0 e^{2t} + c_1 e^t : c_0, c_1 \in \mathbb{R} \} \]

So the solutions are “controlled” by the constant field \( \mathbb{R} \).
Co-analyzability: a form of “controlling”

Suppose $K$ is $\omega$-saturated, $C \subseteq K$ is a definable set.

**Definition**

Let $S \subseteq K^n$ be definable. For $r \in \mathbb{N}$ we say $S$ is **co-analyzable in $r$ steps (relative to $K$ and $C$)** if:

- $(C_0)$ $S$ is co-analyzable in 0 steps iff $S$ is finite;
- $(C_{r+1})$ $S$ is co-analyzable in $r + 1$ steps iff for some definable set $R \subseteq C \times K^n$,
  1. the natural projection $C \times K^n \to K^n$ maps $R$ onto $S$;
  2. for each $c \in C$, the section $R(c) := \{ s \in K^n : (c, s) \in R \}$ above $c$ is co-analyzable in $r$ steps.

We call $S$ **co-analyzable** if $S$ is co-analyzable in $r$ steps for some $r$. 
Consequences of co-analyzability

Fact

Suppose $\mathcal{L}$ is countable, $T$ is complete $\mathcal{L}$-theory such that $T \vdash \exists x C(x)$. Then the following are equivalent for a formula $\varphi(x)$:

1. For some $K \models T$, $\varphi(K)$ is co-analyzable (relative to $C$),
2. For every $K \models T$, $\varphi(K)$ is co-analyzable,
3. For every $K \models T$, if $C_K$ is countable, then so is $\varphi(K)$,
4. For all $K \preceq K^* \models T$, if $C_K = C_{K^*}$, then $\varphi(K) = \varphi(K^*)$.

Moral: (3) and (4) show there is some (possibly complicated) relationship between $C$ and the definable set $\varphi(K)$. 
Theorem (G)

Suppose $K$ is an $H$-field such that

1. $K$ is real closed, $\omega$-free, and newtonian, and
2. $K$ is $\Psi$-closed.

Then for every nonzero differential polynomial $P \in K\{Y\}$, the set

$$Z(P) := \{ y \in K : P(y) = 0 \}$$

is co-analyzable relative to the constant field $C$.

This was known for $T$ (2016, ADH), but new for $T_{\log}$ and other $H$-fields.