Choice, Groups, and Topoi

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If two, you chose from that pair already.

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So "local" is stronger than "global".

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Or (stronger)

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The smallest case not resolved in Mostowski's paper was

$$\left(\forall I\, C(I, \{3, 5, 13\})\right) \implies \left(\forall I\, C(I, 15)\right) ?$$

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Local Theorem: ZF proves

$$\forall I \left(C(I, Z) \implies C(I, n) \right)$$

if and only if, whenever a finite group G acts on an n-element set without fixed-points, then it has proper subgroups $K_1, \ldots, K_r$ (not necessarily distinct or nontrivial) such that $\sum_i |G : K_i| \in Z$.

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The last part of the theorem, "it has proper . . . ," is equivalent to: "G acts without fixed-points on a set whose cardinality is in Z."

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The counterexample group is the group of affine permutations $x \mapsto ax + b$ over the 8-element field.

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An "element" $j : J \to Y$ is in the subobject (represented by) $f : X \to Y$ iff it is $f \circ j'$ for some element j' of X.

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A map $X \sqcup Y \to J$ amounts to a pair $\langle X \to J, Y \to J \rangle$.

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This gives us existential quantification. If $A \subseteq X \times Y$ then its image under the projection p to X is $\{x \in X : (\exists y \in Y)\,(x, y) \in A\}$.

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- For every topos $\mathcal{E}$ of the form M/I, where M is a model of ZF and $I \in M$, if $P(\mathcal{E}/F, z)$ for all $z \in Z$ and all finite objects F of E, then $P(\mathcal{E}, n)$.
- For every topos $\mathcal{E}$ of the form $\mathbf{Set}^G$, where G is a finite group, if $P(\mathcal{E}/F, z)$ for all $z \in Z$ and all finite objects F of E, then $P(\mathcal{E}, n)$.
- For every topos $\mathcal{E}$, if $P(\mathcal{E}/F, z)$ for all $z \in Z$ and all finite, decidable objects F of E, then $P(\mathcal{E}, n)$.

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The required T is the object of bijections (in the internal sense) from n to A.

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