Selectives do not produce Milliken Taylor ultrafilters

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- Ramsey's Theorem is often referred to as a higher dimensional pigeonhole principle, but this does not mean the one dimensional case is without interest.
- Consider van der Waerden's Theorem: For all M and K there is an integer $W(M,K)$ such that if $W(M,K) = \bigcup_{i=1}^{K} P_i$ then there are $i \leq K$ and m and n such that $\{m + jn\}_{j=1}^{M} \subseteq P_i$.
- This can be generalized to higher dimensions, as in the Hales-Jewett Theorem, but this talk will mainly focus on one dimensional generalizations.

Introduction

- A simple consequence of van der Waerden's Theorem is that if $\mathbb{N} = \bigcup_{i=1}^{K} P_i$ then there is $i \leq k$ such that P_i contains arbitrarily long arithmetic progressions.
- This provides the motivation for a conjecture of Graham and Rothschild (1971) that for any partition of $\mathbb{N}$ into finitely many pieces, one of the pieces contains a set that is closed under all sums of distinct members.
- The truth of this conjecture was established by Neil Hindman and is now known as Hindman's Theorem.
- The history of the proof of this theorem may be well known, but its review will motivate the main question of this talk.

- A natural approach to proving Hindman's Theorem would be to proceed inductively to construct $k_0, k_1, \ldots k_n$ and A_n such that all sums of distinct integers from $k_0, k_1, \ldots k_n$ belong to one piece of the given partition and, crucially, A_n is an infinite set from which it is possible to select the next integer k_{n+1}.
- And, indeed, this is how all proofs of Hindman's Theorem proceed, but the technical details in the original proof are daunting.
- A potential stumbling block is correctly choosing the set A_n.
- It would help if the A_n could be selected from an ultrafilter.
- But that ultrafilter needs to enjoy some very specific properties.

- In an early paper (1972) Hindman made the following observation: The Graham and Rothschild Conjecture holds if and only if there is an ultrafilter on $\mathbb{N}$ every member of which contains an infinite subset closed under addition of finite sums.
- A set $A \subseteq \mathbb{N}$ is closed under **addition of finite sums** if $\sum s \in A$ for each non-empty $s \in [A]^{<\aleph_0}$ — so no repetitions.
- In the same paper, he shows that $2^{\aleph_0} = \aleph_1$ implies that there is such an ultrafilter.
- The ultrafilter u constructed by Hindman has the additional property that it is an idempotent.
- Namely $u + u = u$ in the semigroup $(\beta\mathbb{N}, +)$ where ultrafilters are thought of as finitely additive measures and the $+$ operation is the convolution of measures.

- Hindman was later (1974) able to eliminate the use of the Continuum Hypothesis with an elaborate, albeit elementary argument, that seemingly did away with the use of the idempotent.
- Somewhat later, though, Baumgartner (1974) produced a much simpler version of Hindman's technical argument.
- A key idea used by Baumgartner was the notion of a large set, somewhat reminiscent of the construction of Haar measure.
- As noted by Bergelson, the notion of largeness in this context can be traced back to Poincaré's work on celestial mechanics and, when combined with an idempotent ultrafilter, very quickly yields Hindman's Theorem.

- The key realization of Galvin and Glazer (mid 1970's) that a much older and more general theorem of Ellis (from 1958) about idempotents in compact semigroups, could vastly simplify the proof of Hindman's Theorem points to the important role of ultrafilters in this area of Ramsey Theory.
- Even though Hindman's construction using $2^{\aleph_0} = \aleph_1$ was not, ultimately, necessary, it played an important role in the development of the subject and fostered future research.

- For example, van Douwen realized that, assuming $2^{\aleph_0} = \aleph_1$, it is possible to construct an ultrafilter that satisfies a stronger version of the property Hindman established under the same assumptions, namely an ultrafilter with a base consisting of subsets of $\mathbb{N}$ closed under finite sums.
- The difference is worth highlighting: Hindman had asked only that each member of his ultrafilter contain a set closed under finite sums, but van Douwen is asking that this set actually belongs to the ultrafilter.
- These ultrafilters identified by van Douwen are now known as **strongly summable ultrafilters** and the question of whether the Continuum Hypothesis is needed to construct them is also attributed to van Douwen.

Definition

Closely related to Hindman's Theorem is a result about the union operation on the finite subsets of the positive integers. Let $\mathbb{F}$ *denote* $\left\{a \in [\mathbb{N}]^{<\aleph_0} \mid a \neq \varnothing\right\}$. *If* $A \subseteq \mathbb{F}$ *consists of pairwise disjoint sets then*

$$\mathrm{FU}(A) = \left\{\bigcup a \;\middle|\; a \in [A]^{<\aleph_0} \ \& \ a \neq \varnothing\right\}.$$

In analogy with with the strongly summable ultrafilters, it is possible to formulate the following definition. An ultrafilter on $\mathbb{F}$ *will be called a* union ultrafilter *if it has a base consisting of sets of the form* $\mathrm{FU}(A)$.

- However, it turns out that the connection between strongly summable ultrafilters and union ultrafilters goes beyond analogy.
- The mapping from $\mathbb{F}$ to $\mathbb{N}$ sending a to $\sum_{n\in a} 2^n$ sends union ultrafilters to strongly summable ultrafilters.
- It turns out that many of the constructions of union ultrafilters actually produce a stronger property of ultrafilters, known as **ordered-union** ultrafilters, and these will be the focus of this talk.

Introduction

Definition

Define a partial order $<$ on $\mathbb{F}$ by $a < b$ if $\max(a) < \min(b)$. For $A \subseteq \mathbb{F}$ and $\kappa \leq \omega$ let $[A]^{\kappa}_{<}$ denote all sets of the form $\{a_n\}_{n\in\kappa} \subseteq A$ such that $a_n < a_{n+1}$ for all $n \in \kappa$.

Definition

An ultrafilter on $\mathbb{F}$ will be called an ordered-union ultrafilter *if it has a base consisting of sets of the form $\mathrm{FU}(A)$ where $A \in [\mathbb{F}]^{\omega}_{<}$.*

Theorem (Blass and Hindman)

If $2^{\aleph_0} = \aleph_1$ there is a union ultrafilter that is not an ordered-union ultrafilter.

- Blass examined the ordered-union ultrafilters (1985) and considered a further property.
- An ordered-union ultrafilter $\mathcal{U}$ on $\mathbb{F}$ will be called *stable* if it satisfies the following property: Given a sequence of sets $\{A_n\}_{n\in\omega} \subseteq \mathcal{U}$ there is a sequence $\{b_n\}_{n\in\omega} \in [\mathbb{F}]^{\omega}_{<}$ such that for each k there is k^* such that $\mathrm{FU}(\{b_n\}_{n\geq k^*}) \subseteq A_k$ for each $k \in \omega$.
- While stability resembles the P-point property, a union ultrafilter is never a P-point because if $\mathcal{U}$ is a union ultrafilter then $E_k = \{a \in \mathbb{F} \mid k \notin a\} \in \mathcal{U}$ for each $k \in \omega$. But the E_k cannot be diagonalized because if $k \in \bigcup A$ and $\mathrm{FU}(A) \in \mathcal{U}$ then there are infinitely many $a \in A$ such that $k \in A$.

Results from the 1985 paper of Blass

However, Blass showed that the Milliken-Taylor Theorem has the following ultrafilter version.

Theorem

For an ordered-union ultrafilter $\mathcal{H}$ the following are equivalent:

1. $\mathcal{H}$ *is stable;*
2. *if* $[\mathbb{F}]^2_< = \mathcal{A}_0 \cup \mathcal{A}_1$ *then there is* $j \in 2$ *and* $H \in \mathcal{H}$ *such that* $[H]^2_< \subseteq \mathcal{A}_j$;
3. *if* $F : \mathbb{F} \to \omega$ *then there is* $H \in \mathcal{H}$ *such that* F *is canonical on* H.

Definition

For $A \in [\mathbb{F}]^{\omega}_{<}$ define

- $\min(A) = \{\min a \mid a \in A\} = \{\min a \mid a \in FU(A)\}$
- $\max(A) = \{\max a \mid a \in A\} = \{\max a \mid a \in FU(A)\}$

and for any union ultrafilter $\mathcal{U}$ define

- $\min(\mathcal{U}) = \{\min A \mid A \in \mathcal{U}\}$
- $\max(\mathcal{U}) = \{\max A \mid A \in \mathcal{U}\}$.

A routine arguments shows that both $\max(\mathcal{U})$ and $\min(\mathcal{U})$ are ultrafilters. What further properties they have is a more interesting question.

Theorem (Blass and Hindman)

If $\mathcal{U}$ is a union ultrafilter then $\max(\mathcal{U})$ and $\min(\mathcal{U})$ are both P-points.

Using the theorem on equivalence of stability, Blass obtained the following version of the preceding.

Theorem (Blass)

If $\mathcal{U}$ is a stable, ordered-union ultrafilter then $\max(\mathcal{U})$ and $\min(\mathcal{U})$ are both selective ultrafilters. Moreover, $\max(\mathcal{U}) \not\equiv_{RK} \min(\mathcal{U})$.

The preceding theorem follows from the following result, whose proof provides an instructive example of how union ultrafiltrs differ from ordered-union ultrafilters.

Recall that an ultrafilter $\mathcal{U}$ is a Q-point if every finite-to-one function on ω is one-to-one on a set in $\mathcal{U}$. An ultrafilter is selective if and only if it is a P-point and a Q-point.

Theorem (Blass and Hindman)

If $\mathcal{U}$ is an ordered-union ultrafilter then $\max(\mathcal{U})$ and $\min(\mathcal{U})$ are both Q-points (and, hence, selective).

To see this, let $F : \omega \to \omega$. Let

$$Z = \{z \in \mathbb{F} \mid (\forall i \leq \min(z))(\forall j \geq \max(z))\ F(i) \neq F(j)\}$$

and let $\{z_i\}_{i\in\omega}$ be such that either $\mathrm{FU}(\{z_i\}_{i\in\omega}) \subseteq Z$ or $\mathrm{FU}(\{z_i\}_{i\in\omega}) \cap Z = \varnothing$.

The second alternative cannot hold. Why? Since F is finite-to-one there is some k such that $F(i) \neq F(j)$ if $i \leq \min(z_0)$ and $j \leq \min(z_k)$. But then $z_0 \cup z_k \in Z$.

Then $U = \{\min(z_i)\}_{i\in\omega} \in \min(\mathcal{U})$ and, furthermore, if $i < j$ then $\min(z_j) > \max(z_i)$ and so $F(\min(z_j)) \neq F(\min(z_i))$. In other words, F is one-to-one on U.

Note that this argument would not work for a union ultrafilter because it would not be possible to conclude that $\min(z_j) > \max(z_i)$ when $i < j$.

The preceding results havea partial converse.

THEOREM (BLASS)

Assuming $2^{\aleph_0} = \aleph_1$ *for each pair of RK inequivalent selective ulrafilters* $\mathcal{U}$ *and* $\mathcal{V}$ *there is a stable, ordered-union ultrafilter* $\mathcal{W}$ *such that* $\min(\mathcal{W}) = \mathcal{U}$ *and* $\max(\mathcal{W}) = \mathcal{U}$*.*

CONJECTURE (BLASS)

The preceding result cannot be proved without the Continuum Hypothesis (or some other extra axiom).

Construction of the model

In order to construct a model where there are no stable, ordered-union ultrafilters but there are at least two selective ultrafilters the following ingredients are needed.

- A partial for destroying stable, ordered-union ultrafilters.
- An iteration scheme.
- An argument for preserving selective ultarfilters.

Our plan is to use Shelah's construction of a model with a unique P-point (not just a unique selective) as a template.

Construction of the model

- For $s \in \mathbb{F}$ define $s^- = s \setminus \{\max(s)\}$.
- If $s \in \mathbb{F}$ and $k \leq \max(s)$ define $A \subseteq 2^{\mathcal{P}(\max(s))}$ to be (k, s)-large if for every $\sigma : \mathcal{P}(k) \to 2$ there is some $\tau \in A$ such that $\sigma(u) = \tau(u \cup s^-)$ for every $u \subseteq k$.
- Define $\mathbb{T} = \bigcup_{\ell \in \omega} \prod_{k \in \ell} 2^{\mathcal{P}(k)}$.
- If $\mathcal{H}$ is a stable, ordered-union ultrafilter define $\mathbb{P}(\mathcal{H})$ to consist of all trees $T \subseteq \mathbb{T}$ such that for each $k \in \omega$ the set of $s \in \mathbb{F}$ such that $k \leq \max(s)$ and

$$(\forall t \in 2^{\mathcal{P}(\max(s))} \cap T) \ \{\tau \mid t^\frown \tau \in T\} \text{ is } (k, s) \text{ large}$$

belongs to $\mathcal{H}$.

Construction of the model

- What does $\mathbb{P}(\mathcal{H})$ accomplish?
- If $G \subseteq \mathbb{P}(\mathcal{H})$ be generic then $\bigcup\bigcap G$ a function $g \in \prod_{n\in\omega} 2^{\mathcal{P}(n)}$.
- This can be used to define a partition P_G of $\mathbb{F}$ by $P_G(s) = g(\max(s))(s^-)$.
- It is shown that if $\mathrm{FU}(\{a_n\}_{n\in\omega}) \subseteq P_G^{-1}(j)$ for $j \in 2$ then $\mathrm{FU}(\{a_n\}_{n\in\omega}) \in \mathcal{H}^*$.
- And this remains true even after further forcing of a specified type.

Construction of the model

A simpler version of this forcing plays a role in the eventual argument, but is also serves as a useful prototype.

- If $k < m \in \omega$ define $A \subseteq 2^m$ to be k-large if for every $\sigma : k \to 2$ there is some $\tau \in A$ such that $\sigma \subseteq \tau$.
- Define $\mathbb{T} = \bigcup_{\ell\in\omega} \prod_{k\in\ell} 2^k$.
- If $\mathcal{S}$ is a selective ultrafilter define $\mathbb{P}(\mathcal{S})$ to consist of all trees $T \subseteq \mathbb{T}$ such that for each $k \in \omega$ the set of $m > k$ such that

$$(\forall t \in 2^m \cap T)\ \{\tau \mid t^\frown\tau \in T\} \text{ is } k\text{-large}$$

  belongs to $\mathcal{S}$.
- If $G \subseteq \mathbb{P}(\mathcal{S})$ is generic it adds $P_G : [\omega]^2 \to 2$.

- To see that if $\mathcal{S}$ is selective and $G \subseteq \mathbb{P}(\mathcal{S})$ is generic then any P_G homogeneous set is in $\mathcal{S}^*$ suppose that $T \Vdash_{\mathbb{P}(\mathcal{S})}$ "$P_G([\dot{A}]^2) = 0$".
- It can be shown that there is $T^* \subseteq T$ and $S \in \mathcal{S}$ and $\psi : S \to \omega$ such that:
    - $T^* \Vdash_{\mathbb{P}(\mathcal{S})}$ "$(\forall n \in \omega)\ [n, \psi(n)) \cap \dot{A} \neq \varnothing$";
    - if $t \in T$ and $|t| \in S \setminus \psi(n)$ then the successors of t are $\psi(n)$-large;
    - $S \cap (n, \psi(n)) = \varnothing$.
- Define T^{**} so that if $n < m$ are consecutive elements of S and $t \in T^*$ and $|t| = m$ the successor of t in T^{**} are all those τ such that $\tau(j) = 1$ if $n \leq j < \psi(n)$. The remaining set of successors is no longer $\psi(n)$-large, but is still n-large.

So, starting with $\{D_\xi\}_{\xi\in\omega_2}$ satisfying $\diamondsuit_{\omega_2,\mathrm{cof}(\omega_1)}$ and constructing an iteration $\mathbb{P}_\xi$ for $\xi \in \omega_2$ such that

$(\forall\alpha)$ if $1 \Vdash_{\mathbb{P}_\alpha}$ "D_α codes an ultrafilter $\mathcal{D}_\alpha$" then $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \mathbb{P}(\mathcal{D}_\alpha)$

will achieve that all stable, ordered-union ultrafilters are destroyed.

But why are any selective ultrafilters preserved?

Construction of the model

Definition

Suppose $s \in \mathbb{F}$. *Define* $I(s) = \{k \in \omega \mid \min(s) \leq k \leq \max(s)\}$. *For* $X \in \mathrm{FU}(\mathbb{F})$ *define* $N(X) = \bigcup_{i\in\omega} I(X(i)) \subseteq \omega$.

For a stable, ordered-union ultrafilter $\mathcal{H}$ *define* $\mathcal{C}_0(\mathcal{H})$ *to be the set of all selectives* $\mathcal{U}$ *such that: for every* $X \in [\mathbb{F}]^\omega_<$ *such that* $\mathrm{FU}(X) \in \mathcal{H}$ *there is* $Y \in [\mathbb{F}]^\omega_<$ *such that* $\mathrm{FU}(Y) \in \mathcal{H}$, $\mathrm{FU}(Y) \subseteq \mathrm{FU}(X)$ *and* $N(Y) \notin \mathcal{U}$.

Lemma

If $\mathcal{H}$ *is a stable, ordered-union ultrafilter then* $\mathcal{H}_{\min} \notin \mathcal{C}_0(\mathcal{H})$ *and* $\mathcal{H}_{\max} \notin \mathcal{C}_0(\mathcal{H})$.

Lemma

If $\mathcal{H}$ is a stable, ordered-union ultrafilter and $\mathcal{U}$ is a selective ultrafilter such that $\mathcal{U} \equiv_{RK} \max(\mathcal{H})$ then $\mathbb{P}(\mathcal{H})$ preserves $\mathcal{U}$.

Definition

Define $\mathcal{C}_1(\mathcal{H})$ to be the collection of all selective $\mathcal{U}$ such that: $\mathcal{U} \not\equiv_{RK} \mathcal{V}$ for every selective $\mathcal{V}$ not in $\mathcal{C}_0(\mathcal{H})$

Theorem

If $\mathcal{V}$ is a selective and $\mathcal{V} \notin \mathcal{C}_1(\mathcal{H})$ then $\mathbb{P}(\mathcal{H})$ preserves $\mathcal{V}$.

The preservation of selectives depends on a number of results about various games. For example, the following is well known. If $\mathcal{U}$ is an ultrafilter then the game $\Game^{\text{select}}(\mathcal{U})$ is played as follows:

- In the n^{th} Inning **Player 1** plays $A_n \in \mathcal{U}$
- **Player 2** then plays $k_n \in A_n$
- **Player 2** wins if $\{k_n\}_{n\in\omega} \in \mathcal{U}$.

If $\mathcal{H}$ is an ultrafilter on $\mathbb{F}$ then the game $\Game^{\text{stable}}(\mathcal{H})$ is played as follows:

- In the n^{th} Inning **Player 1** plays $A_n \in \mathcal{H}$
- **Player 2** then plays $s_n \in A_n$ such that $s_{n-1} < s_n$
- **Player 2** wins if $\mathrm{FU}(\{\mathrm{s_n}\}_{\mathrm{n}\in\omega}) \in \mathcal{H}$.

Given ultrafilters $\mathcal{U}$ on ω and and $\mathcal{H}$ on $\mathbb{F}$ the game $\Game^{\text{select,stable}}(\mathcal{U},\mathcal{H})$ is played:

- like $\Game^{\text{select}}(\mathcal{U})$ in even innings
- like $\Game^{\text{stable}}(\mathcal{H})$ in odd innings
- to win, **Player 2** must win both games.

The preservation of certain selectives depends on determining for which pairs of ultrafilters **Player 1** has no winning strategy in the associated game.

Theorem

For any cardinal κ such that $0 \leq \kappa \leq \aleph_2$ it is consistent that there are κ RK inequivalent selective ultrafilters, but no stable, ordered-union ultrafilters.

Questions and Remarks

- Forcing with $[\omega]^{\aleph_0}$ ordered by almost inclusion adds a selective ultrafilter and forcing with $[\mathbb{F}]^{\omega}_{<}$ ordered by almost refinement adds a stable ordered-union ultrafilter.
- Todorcevic: In the presence of large cardinals, the selective ultrafilters are precisely those ultrafilters on ω that are generic over $\mathbf{L}(\mathbb{R})$ for $[\omega]^{\aleph_0}$ partially ordered by almost inclusion.
- Similarly, in the presence of large cardinals, the stable, ordered-union ultrafilters are those ultrafilters on $\mathbb{F}$ that are generated from a generic filter over $\mathbf{L}(\mathbb{R})$ for $[\mathbb{F}]^{\omega}_{<}$ partially ordered by almost refinement.

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Questions and Remarks

- The space corresponding to Ramsey's theorem is the *Ellentuck space* and the one corresponding to Hindman's theorem and the Milliken-Taylor theorem has a corresponding Ramsey space.
- Hence, in the presence of large cardinals, the selective ultrafilters are the generic ultrafilters corresponding to the Ellentuck space, while the stable, ordered-union ultrafilters are the generic ultrafilters corresponding to the Milliken-Taylor space.
- The preceding theorem can be interpreted as saying that generics on a lower Ramsey space need not pull back to the higher Ramsey space, even if there are many of them.

Questions and Remarks

Question

For which Ramsey spaces do generics on one imply generics on the other?

Question

For example: Do generics one the Milliken-Taylor space yield generics on the Hales-Jewett space? What about the Gowers Theorem?

Question (Smythe)

Is every ordered-union ultrafilter stable?

Question

Does the existence of two RK inequivalent P-points imply the existence of a union ultrafilter? What about selectives?