Menger spaces everywhere

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Menger spaces and relatives

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Corollary

Luzin sets are not Hurewicz.

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Fact. There exists a $\mathfrak{d}$-concentrate set.

Proof. Fix a dominating $\{d_\alpha : \alpha < \mathfrak{d}\} \subset \omega^\omega$ and inductively construct $S = \{s_\alpha : \alpha < \mathfrak{d}\} \subset \omega^{\uparrow\omega}$ such that $s_\alpha \not\leq^* d_\beta$ for all $\beta \leq \alpha$.

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Proof. Fix an unbounded $B = \{b_\alpha : \alpha < \mathfrak{b}\} \subset \omega^\omega$ such that $b_\beta \leq^* b_\alpha$ for all $\beta \leq \alpha$. B is $\mathfrak{b}$-concentrated on Q. $\square$

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Given $x, y \in \omega^\omega$, $x \leq^* y$ means $\{n : x(n) \leq y(n)\}$ is cofinite. $\mathfrak{b}$ is the minimal cardinality of an unbounded subset of ω^ω. $\mathfrak{d}$ is the minimal cardinality of an unbounded subset of ω^ω.

$|X| < \mathfrak{b} \rightarrow X$ is Hurewicz. $\mathfrak{b}$- Sierpinski sets are Hurewicz.
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Fact. There exists a $\mathfrak{d}$-concentrate set.

Proof. Fix a dominating $\{d_\alpha : \alpha < \mathfrak{d}\} \subset \omega^\omega$ and inductively construct $S = \{s_\alpha : \alpha < \mathfrak{d}\} \subset \omega^{\uparrow\omega}$ such that $s_\alpha \not\leq^* d_\beta$ for all $\beta \leq \alpha$. Viewed as a subspace of $(\omega+1)^{\uparrow\omega}$, S is $\mathfrak{d}$-concentrated on $Q = \{x \in (\omega+1)^{\uparrow\omega} : x$ is eventually $\omega\}$. □

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Proposition

$\mathrm{add}(\mathrm{Menger}) \in [\min\{\mathfrak{b}, \mathfrak{g}\}, \mathrm{cf}(\mathfrak{d})]$ □.

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$\{U_y : y \in X\}$ is an open cover of X is V, so it contains a countable subcover $\{U_{y_n} : n \in \omega\}$, as witnessed by $\{p_n : n \in \omega\} \subset Fn(\mu, 2)$. By elementarity, we can assume $\{U_{y_n} : n \in \omega\}, \{p_n : n \in \omega\} \in M$, and hence $\{U_{y_n} : n \in \omega\} \cup \{p_n : n \in \omega\} \subset M$. Pick n such that $x \in U_{y_n}$ and note that $p_n \in D_x$.

Let G be $Fn(\mu, 2)$-generic. Then $H := G \cap M$ is $Fn(\mu, 2) \cap M$ generic.
$\dot{\mathcal{U}}^G \cap M$ covers X: given $x \in X$, find $p \in D_x \cap H$ and $U \in \tau \cap M$ witnessing this, and note that $p \in G$ and $p \Vdash U \in \dot{\mathcal{U}}$, and hence $x \in U \in \dot{\mathcal{U}}^G$. $\square$

Theorem (Essentially A. Dow)

Let (X, τ) *be a Lindelöf space. Then* X *is Menger in* $V^{Fn(\mu,2)}$*.*

Proof. Two steps. 1. X remains Lindelöf. 2. X becomes Menger.

Proof of 1. Let $\dot{\mathcal{U}}$ be a $Fn(\mu, 2)$-name for an open cover of X by ground model open sets and $M \prec H(\theta)$ be such that $\dot{\mathcal{U}}, X, \mu, ... \in M$. Given any $x \in X$, consider
$D_x = \{p \in Fn(\mu, 2) \cap M : \exists U \in \tau \cap M \ (x \in U \wedge p \Vdash U \in \dot{\mathcal{U}}).\}$
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Menger game

Game associated to Menger's property:

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Theorem (Telgarsky 197?)

Let X be a hereditarily Lindelöf regular space. If II has a winning strategy in the Menger game on X, then X is σ-compact.

Theorem (Hurewicz 192?)

X is Menger if and only if I has no winning strategy in the Menger game on X.

Proof. Sp-se X is Menger. Given a strategy F of I, we'll construct a play won by II, in which I uses F.

The non-existence of a winning strategy for I

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$$\Big\langle \mathcal{U}_\emptyset, U_{\langle n_0\rangle}; \; F\langle U_{\langle n_0\rangle}\rangle = \mathcal{U}_{\langle n_0\rangle}, U_{\langle n_0,n_1\rangle}; \; \dots, \\ F\Big\langle U_{\langle n_0\rangle}, \dots, U_{\langle n_0,\dots,n_{m-1}\rangle}\Big\rangle = \mathcal{U}_{\langle n_0,\dots n_{m-1}\rangle}, U_{\langle n_0,\dots n_{m-1},n_m\rangle} = U_\sigma\Big\rangle$$

in which I uses F, and the next response of I is $\mathcal{U}_\sigma = \{U_{\sigma\,\hat{}\,k} : k \in \omega\}$ with $U_{\sigma\,\hat{}\,k} \subset U_{\sigma\,\hat{}\,\langle k+1\rangle}$.

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$$\Big\langle \mathcal{U}_\emptyset, U_{\langle n_0\rangle}; F\langle U_{\langle n_0\rangle}\rangle = \mathcal{U}_{\langle n_0\rangle}, U_{\langle n_0,n_1\rangle}; \ldots, F\Big\langle U_{\langle n_0\rangle}, \ldots, U_{\langle n_0,\ldots,n_{m-1}\rangle}\Big\rangle = \mathcal{U}_{\langle n_0,\ldots n_{m-1}\rangle}, U_{\langle n_0,\ldots n_{m-1},n_m\rangle} = U_\sigma\Big\rangle$$

in which I uses F, and the next response of I is $\mathcal{U}_\sigma = \{U_{\sigma\,\hat{}\,k} : k \in \omega\}$ with $U_{\sigma\,\hat{}\,k} \subset U_{\sigma\,\hat{}\,\langle k+1\rangle}$. Wlog, $U_\sigma = U_{\sigma\,\hat{}\,0}$.

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A space (X,τ) is called a *D-space*, if for every $f : X\to\tau$ such that $x\in f(x)$ for all x, there exists a closed discrete $D\subset X$ such that $X=\bigcup_{x\in D} f(x)$.

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Problem

Is every regular Lindelöf space a D-space?

A solution :)

Let X be a regular Lindelöf space and $f : X \to \tau$ a neighbourhood assignment as above. Since X is paracompact, there exists a refinement $\mathcal{U}$ of $f[X] = \{f(x) : x \in X\}$ which is locally finite.

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I.e., for every $U \in \mathcal{U}$ there exists $x_U \in X$ such that $U \subset f(x_U) \in \tau$ and every $x \in X$ has a neighbourhood $O(x)$ which intersects only finitely many $U \in U$.

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Is there maybe any problem in this argument?

Menger spaces are D-spaces (Aurichi 2010).

Let f be a neighbourhood assignment. Consider the following strategy of I in the Menger game on X. $\mathcal{U}_\emptyset = \{f(x) : x \in X\}$. Suppose that II replies with $\{f(x) : x \in F_0\}$ for some $F_0 \in [X]^{<\omega}$.

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There is a play lost by I, which yields a sequence $\langle U_n = \bigcup_{x \in F_n} f(x) : n \in \omega\rangle$ covering X s.t. $F_{n+1} \subset X \setminus \bigcup_{i \leq n} U_n$. $\bigcup_{n \in \omega} F_n$ is a closed discrete kernel of f. $\square$

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Applications: killing mad families, making the ground model reals not splitting, etc.

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There is a combinatorial characterization of Canjar filters by Hrusak and Minami in terms of the filter $\mathcal{F}^{<\omega}$ on $[\omega]^{<\omega}$ generated by $\{[F]^{<\omega} : F \in \mathcal{F}\}$.

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If an ultrafilter $\mathcal{F}$ is Canjar, then it is a P-filter and there is no monotone surjection $\varphi : \omega \to \omega$ such that $\varphi(\mathcal{F})$ is rapid.

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Recall that a filter $\mathcal{F}$ is a *coherent strong P^+-filter* if for every sequence $\langle \mathcal{C}_n : n \in \omega \rangle$ of compact subsets of $\mathcal{F}^+$ there exists an increasing sequence $\langle k_n : n \in \omega \rangle$ of integers such that if $X_n \in \mathcal{C}_n$ for all n
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Strong P^+-filters are defined by removing the coherence requirement.

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$\mathbb{M}_{\mathcal{F}}$ *is Canjar iff* $\mathcal{F}$ *has the Menger covering property as a subspace of* $\mathcal{P}(\omega)$*.* $\square$

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Corollary

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Some corollaries

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Answers a question of Hrusak and Minami.

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A filter $\mathcal{F}$ is Canjar iff it is a strong P^+-filter. $\square$

Theorem (Guzman-Hrusak-Martinez 2013)

A filter $\mathcal{F}$ is Canjar iff it is a coherent strong P^+-filter. $\square$

Recall that a filter $\mathcal{F}$ is a *coherent strong P^+-filter* if for every sequence $\langle \mathcal{C}_n : n \in \omega \rangle$ of compact subsets of $\mathcal{F}^+$ there exists an increasing sequence $\langle k_n : n \in \omega \rangle$ of integers such that if $X_n \in \mathcal{C}_n$ for all n

and $X_m \cap [k_n, k_{n+1}) \subset X_n \cap [k_n, k_{n+1})$ for $n < m$,

then $\bigcup_{n\in\omega}(X_n \cap [k_n, k_{n+1})) \in \mathcal{F}^+$.

Strong P^+-filters are defined by removing the coherence requirement.

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Proof of "$\mathcal{F}$ is Hurewicz iff $\mathbb{M}_{\mathcal{F}}$ is almost ω^ω-bounding".

Suppose that $\mathcal{F}$ is Hurewicz, but there exists an unbounded $X \subset \omega^\omega$, $X \in V$, and an $\mathbb{M}_{\mathcal{F}}$-name $\dot{g}$ for a function dominating X (as forced by $1_{\mathbb{M}_{\mathcal{F}}}$).

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Fix $x \in X$ and $l \in \omega$ such that for every $m \geq l$ there exists $s_m \in \mathcal{T}_m$ such that $F^x \in \uparrow s_m$.

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Let G be the generic pseudointersection of $\mathcal{F}$ added by $\mathbb{M}_{\mathcal{F}}$.

Fix $x \in X$ and $l \in \omega$ such that for every $m \geq l$ there exists $s_m \in \mathcal{T}_m$ such that $F^x \in \uparrow s_m$. Pick any $m \geq n_*, l$. Since $\langle s_*, F^x \rangle \Vdash x(m) < \dot{g}(m)$, $\langle s_* \cup s_m, F_{s_m} \rangle \Vdash \dot{g}(m) \leq f(m)$, and these two conditions are compatible, it follows that $x(m) < f(m)$.

Now suppose that $\mathcal{F}$ is not Hurewicz as witnessed by a sequence $\langle \mathcal{U}_n \colon n \in \omega \rangle$ of covers of $\mathcal{F}$ by sets open in $\mathcal{P}(\omega)$. Wlog $\mathcal{U}_n = \{\uparrow q_m(n) \colon m \in \omega\}$, where $q_m(n) \in [\omega]^{<\omega}$. For every $F \in \mathcal{F}$ consider the function $x_F \in \omega^\omega$, $x_F(n) = \min\{m \colon F \in \uparrow q_m(n)\}$. $X = \{x_F \colon F \in \mathcal{F}\}$ is unbounded.

Let G be the generic pseudointersection of $\mathcal{F}$ added by $\mathbb{M}_{\mathcal{F}}$. For every n there exists $g(n)$ such that $G \setminus n \in \uparrow q_{g(n)}(n)$.

Fix $x \in X$ and $l \in \omega$ such that for every $m \geq l$ there exists $s_m \in \mathcal{T}_m$ such that $F^x \in\uparrow s_m$. Pick any $m \geq n_*, l$. Since $\langle s_*, F^x\rangle \Vdash x(m) < \dot{g}(m)$, $\langle s_* \cup s_m, F_{s_m}\rangle \Vdash \dot{g}(m) \leq f(m)$, and these two conditions are compatible, it follows that $x(m) < f(m)$.

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Now suppose that $\mathcal{F}$ is not Hurewicz as witnessed by a sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of covers of $\mathcal{F}$ by sets open in $\mathcal{P}(\omega)$. Wlog $\mathcal{U}_n = \{\uparrow q_m(n) : m \in \omega\}$, where $q_m(n) \in [\omega]^{<\omega}$. For every $F \in \mathcal{F}$ consider the function $x_F \in \omega^\omega$, $x_F(n) = \min\{m : F \in \uparrow q_m(n)\}$. $X = \{x_F : F \in \mathcal{F}\}$ is unbounded.

Let G be the generic pseudointersection of $\mathcal{F}$ added by $\mathbb{M}_{\mathcal{F}}$. For every n there exists $g(n)$ such that $G \setminus n \in \uparrow q_{g(n)}(n)$. Fix $F \in \mathcal{F}$ and find n such that $G \setminus n \subset F$. Then $G \setminus n \in \uparrow q_{g(n)}(n)$ yields $F \in \uparrow q_{g(n)}(n)$, which implies $x_F(n) \leq g(n)$. Thus $g \in \omega^\omega$ is dominating X, and therefore $\mathbb{M}_{\mathcal{F}}$ fails to preserve ground model unbounded sets. $\square$

Questions

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Let $\mathcal{A} \subset [\omega]^\omega$ be a mad family. Is there a Hurewicz filter $\mathcal{F}$ containing $\mathcal{F}(\mathcal{A})$?

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A positive answer would give the consistency of $\mathfrak{s} = \mathfrak{b} = \omega_1 < \mathfrak{a}$.

 

Thanks to >50 Nations helping Ukraine to survive!

The last slide

Thank you for your attention.