

Menger spaces everywhere

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Menger spaces and relatives

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Corollary

Luzin sets are not Hurewicz.

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Fact. There exists a \mathfrak{d} -concentrate set.

Proof. Fix a dominating $\{d_\alpha : \alpha < \mathfrak{d}\} \subset \omega^\omega$ and inductively construct $S = \{s_\alpha : \alpha < \mathfrak{d}\} \subset \omega^{\uparrow\omega}$ such that $s_\alpha \not\leq^* d_\beta$ for all $\beta \leq \alpha$.

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Proof. Fix an unbounded $B = \{b_\alpha : \alpha < \mathfrak{b}\} \subset \omega^\omega$ such that $b_\beta \leq^* b_\alpha$ for all $\beta \leq \alpha$. B is \mathfrak{b} -concentrated on Q . \square

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Given $x, y \in \omega^\omega$, $x \leq^* y$ means $\{n : x(n) \leq y(n)\}$ is cofinite. \mathfrak{b} is the minimal cardinality of an unbounded subset of ω^ω . \mathfrak{d} is the minimal cardinality of an unbounded subset of ω^ω .

$|X| < \mathfrak{b} \rightarrow X$ is Hurewicz. \mathfrak{b} -Sierpinski sets are Hurewicz.

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Fact. There exists a \mathfrak{d} -concentrate set.

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Proposition

$\text{add}(\text{Menger}) \in [\min\{\mathfrak{b}, \mathfrak{g}\}, \text{cf}(\mathfrak{d})]$

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Fact. (CH.) There are two Sierpinski (hence Hurewicz) sets S_0, S_1 whose product is not Menger.

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Menger spaces and forcing

Theorem (Essentially A. Dow)

Let (X, τ) be a Lindelöf space. Then X is Menger in $V^{Fn(\mu, 2)}$.

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Let G be $Fn(\mu, 2)$ -generic. Then $H := G \cap M$ is $Fn(\mu, 2) \cap M$ generic.

$\dot{U}^G \cap M$ covers X : given $x \in X$, find $p \in D_x \cap H$ and $U \in \tau \cap M$ witnessing this, and note that $p \in G$ and $p \Vdash U \in \dot{U}$, and hence $x \in U \in \dot{U}^G$.

Theorem (Essentially A. Dow)

Let (X, τ) be a Lindelöf space. Then X is Menger in $V^{Fn(\mu, 2)}$.

Proof. Two steps. 1. X remains Lindelöf. 2. X becomes Menger.

Proof of 1. Let \dot{U} be a $Fn(\mu, 2)$ -name for an open cover of X by ground model open sets and $M \prec H(\theta)$ be such that $\dot{U}, X, \mu, \dots \in M$. Given any $x \in X$, consider

$$D_x = \{p \in Fn(\mu, 2) \cap M : \exists U \in \tau \cap M (x \in U \wedge p \Vdash U \in \dot{U})\}$$

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Theorem (Telgarsky 197?)

Let X be a hereditarily Lindelöf regular space. If II has a winning strategy in the Menger game on X , then X is σ -compact.

The non-existence of a winning strategy for I

Theorem (Hurewicz 192?)

X is Menger if and only if I has no winning strategy in the Menger game on X.

Proof. Sp-se X is Menger. Given a strategy F of I, we'll construct a play won by II, in which I uses F .

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$U_\sigma = \{U_{\sigma \hat{\ } k} : k \in \omega\}$ with $U_{\sigma \hat{\ } k} \subset U_{\sigma \hat{\ } \langle k+1 \rangle}$. Wlog, $U_\sigma = U_{\sigma \hat{\ } 0}$.

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A space (X, τ) is called a ***D-space***, if for every $f : X \rightarrow \tau$ such that $x \in f(x)$ for all x , there exists a closed discrete $D \subset X$ such that $X = \bigcup_{x \in D} f(x)$.

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Problem

Is every regular Lindelöf space a D-space?

A solution :)

Let X be a regular Lindelöf space and $f : X \rightarrow \tau$ a neighbourhood assignment as above. Since X is paracompact, there exists a refinement \mathcal{U} of $f[X] = \{f(x) : x \in X\}$ which is locally finite.

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I.e., for every $U \in \mathcal{U}$ there exists $x_U \in X$ such that $U \subset f(x_U) \in \tau$ and every $x \in X$ has a neighbourhood $O(x)$ which intersects only finitely many $U \in \mathcal{U}$.

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Since any selection of \mathcal{U} gives a closed discrete subset, $\{x_U : U \in \mathcal{U}\}$ is as required.

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Is there maybe any problem in this argument?

Menger spaces are D -spaces (Aurichi 2010).

Let f be a neighbourhood assignment. Consider the following strategy of I in the Menger game on X . $\mathcal{U}_0 = \{f(x) : x \in X\}$. Suppose that II replies with $\{f(x) : x \in F_0\}$ for some $F_0 \in [X]^{<\omega}$.

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 $\bigcup_{n \in \omega} F_n$ is a closed discrete kernel of f . □

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Applications: killing mad families, making the ground model reals not splitting, etc.

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There is a combinatorial characterization of Canjar filters by Hrusak and Minami in terms of the filter $\mathcal{F}^{<\omega}$ on $[\omega]^{<\omega}$ generated by $\{[F]^{<\omega} : F \in \mathcal{F}\}$.

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Strong P^+ -filters are defined by removing the coherence requirement.

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A filter \mathcal{F} is Canjar iff it is a strong P^+ -filter.

Theorem (Guzman-Hrusak-Martinez 2013)

A filter \mathcal{F} is Canjar iff it is a *coherent* strong P^+ -filter. □

Recall that a filter \mathcal{F} is a *coherent* strong P^+ -filter if for every sequence $\langle \mathcal{C}_n : n \in \omega \rangle$ of compact subsets of \mathcal{F}^+ there exists an increasing sequence $\langle k_n : n \in \omega \rangle$ of integers such that if $X_n \in \mathcal{C}_n$ for all n

and $X_m \cap [k_n, k_{n+1}) \subset X_n \cap [k_n, k_{n+1})$ for $n < m$,
then $\bigcup_{n \in \omega} (X_n \cap [k_n, k_{n+1})) \in \mathcal{F}^+$.

Strong P^+ -filters are defined by removing the coherence requirement.

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For $n \in \omega$ and $q \subset n$ we set $[n, q] := \{A \in \mathcal{P}(\omega) : A \cap n = q\}$.

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Proof of “ \mathcal{F} is Hurewicz iff $\mathbb{M}_{\mathcal{F}}$ is almost ω^ω -bounding”.

Suppose that \mathcal{F} is Hurewicz, but there exists an unbounded $X \subset \omega^\omega$, $X \in V$, and an $\mathbb{M}_{\mathcal{F}}$ -name \dot{g} for a function dominating X (as forced by $1_{\mathbb{M}_{\mathcal{F}}}$).

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Let G be the generic pseudointersection of \mathcal{F} added by $\mathbb{M}_{\mathcal{F}}$. For every n there exists $g(n)$ such that $G \setminus n \in \uparrow q_{g(n)}(n)$.

Fix $x \in X$ and $l \in \omega$ such that for every $m \geq l$ there exists $s_m \in \mathcal{T}_m$ such that $F^x \in \uparrow s_m$. Pick any $m \geq n_*, l$. Since $\langle s_*, F^x \rangle \Vdash x(m) < \dot{g}(m)$, $\langle s_* \cup s_m, F_{s_m} \rangle \Vdash \dot{g}(m) \leq f(m)$, and these two conditions are compatible, it follows that $x(m) < f(m)$.

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Let G be the generic pseudointersection of \mathcal{F} added by $\mathbb{M}_{\mathcal{F}}$. For every n there exists $g(n)$ such that $G \setminus n \in \uparrow q_{g(n)}(n)$. Fix $F \in \mathcal{F}$ and find n such that $G \setminus n \subset F$. Then $G \setminus n \in \uparrow q_{g(n)}(n)$ yields $F \in \uparrow q_{g(n)}(n)$, which implies $x_F(n) \leq g(n)$. Thus $g \in \omega^\omega$ is dominating X , and therefore $\mathbb{M}_{\mathcal{F}}$ fails to preserve ground model unbounded sets. □

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A positive answer would give the consistency of $\mathfrak{s} = \mathfrak{b} = \omega_1 < \mathfrak{a}$.







Thanks to >50 Nations helping Ukraine to survive!

Thank you for your attention.