Tutorial on Generalized Descrptive Set Theory

Miguel Moreno
University of Vienna

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This notes are based on a series of talks given at the Set Theory seminar of University of Vienna. This notes are intended to be as close as possible to the transcripts of those seminar session. Due to the nature of the seminar and the questions from the audience, some proofs were split into different sessions in order to give examples and clear answers to the questions from the audience.

1 Descriptive Set Theory (preliminaries)

Definition 1.1 (The Baire space $\mathbf{B}$). *The Baire space is the set ω^ω endowed with the following topology. For every $\eta \in \omega^n$ for some n, define the following basic open set*

$$N_\eta = \{f \in \omega^\omega \mid \eta \subseteq f\}$$

the open sets are of the form $\bigcup X$ where X is a collection of basic open sets.

This topology is metrizable, let $d(f,g) = \frac{1}{n+1}$ where n is the least natural number that satisfies $f(n) \neq g(n)$, in case it does not exist then $f = g$ and $d(f,g) = 0$.

Definition 1.2 (The Cantor space $\mathbf{C}$). *The cantor space is the set 2^ω with the relative subspace topology.*

Definition 1.3 (Borel class). *Let $S \in \{\mathbf{B}, \mathbf{C}\}$. The class $Borel(S)$ of all Borel sets in S is the least collection of subsets of S which contains all open sets and is closed under complements, countable unions and countable intersections.*

Definition 1.4 (Borel hierarchy). *Let $S \in \{\mathbf{B}, \mathbf{C}\}$. Define the classes $\Sigma_\alpha(S)$ and $\Pi_\alpha(S)$, $\alpha < \omega_1$, as follows.*

1. *$\Sigma_1(S)$ is the class of open sets.*
2. *$\Pi_1(S)$ is the class of closed sets.*
3. *For all $\alpha > 1$, $\Sigma_\alpha(S)$ is the class of of all countable unions of sets from $\bigcup_{\beta<\alpha} \Pi_\beta(S)$.*
4. *For all $\alpha > 1$, $\Pi_\alpha(S)$ is the class of of all countable unions of sets from $\bigcup_{\beta<\alpha} \Sigma_\beta(S)$.*

Exercise 1.1.
1. *For all $n < \omega$ and all $\eta \in \omega^n$ the set N_η is closed.*
2. *For all $\beta < \alpha < \omega_1$, $\Sigma_\beta(\mathbf{B}) \subseteq \Sigma_\alpha \mathbf{B}$.*
3. *$Borel(\mathbf{B}) = \bigcup_{0<\alpha<\omega_1} \Sigma_\alpha(\mathbf{B})$.*
4. *$\mid Borel(\mathbf{B}) \mid = 2^\omega$.*
5. *There are subsets of $\mathbf{B}$ that are not Borel.*

Definition 1.5. *Let $S \in \{\mathbf{B}, \mathbf{C}\}$. We say that $A \subseteq S$ is co-meager, if it contains a countable intersection of open and dense subsetes of S. A subset of S is meager, if the cmplement of it is co-meager.*

Definition 1.6. *Let $S \in \{\mathbf{B}, \mathbf{C}\}$. We say that $X \subseteq S$ has the property of Baire (PB) if there is an open set $U \subseteq S$ such that $X \Delta U$ is meager.*

Lemma 1.7. *Every Borel subset of $\mathbf{B}$ has the property of Baire.*

Exercise 1.2. *Prove Lemma 1.7. (Hint: prove that X has the PB if and only if $\mathbf{B} \backslash X$ has the PB.)*

Definition 1.8 ($Borel^*$-code). *Let X be a non-emprty set.*

1. *A subset $T \subset X^{<\omega}$ is a tree if for all $f \in T$ with $n = dom(f) > 0$ and for all $m < n$, $f \restriction m \in T$.*

2. *A non-empty tree $T \subset X^{<\omega}$ is called an ω-tree if the following holds:*

   (a) *If $f : n \to X$ is in T and $n > 0$, then for all $x \in X$, $f \restriction (n-1) \cup \{(n-1, x)\} \in T$.*

   (b) *There is no $f : \omega \to X$ such that for all $n < \omega$, $f \restriction n \in T$.*

3. *We order T by $\subseteq$. The maximal elements of T are called leaves and the set of leaves is denoted by $L(T)$. The least element of T is called root ($\emptyset$). For every $f \in T$ that is not the root, we denote by f^- the immediate predecessor of f in T. We call node every element that is not a leaf.*

4. *A Borel*-code is a pair (T, π), where $T \subseteq (\omega \times \omega)^{<\omega}$ is an ω-tree and π is a function from $L(T)$ to the basic open sets of $\mathbf{B}$.*

5. *Given a Borel*-code (T, π) and $\eta \in \mathbf{B}$, we define the game $GB^*(\eta, (T, \pi))$ as follows. The game $GB^*(\eta, (T, \pi))$ is played by two players, **I** and **II**. In each move $0 \leq n < \omega$ the function $f_n : n+1 \to (\omega \times \omega)$ from T is chosen as follows: Suppose $f_{n-1} \in T$ is chosen, in case $n = 0$, $f_{-1} = \emptyset$. If f_{n-1} is not a leaf, then **I** choose some $i < \omega$ and then **II** choose some $j < \omega$. This determines $f_n = f_{n-1} \cup \{(n, (i, j))\}$. If f_{n-1} is a leaf, then the game ends and **II** wins if $\eta \in \pi(f_{n-1})$.*

6. *A function $W : \omega^{<\omega} \to \omega$ is a winning strategy of **II** in $GB^*(\eta, (T, \pi))$, if **II** wins by choosing $W(i_0, \ldots, i_n)$ on the move n, where $i_0, \ldots, i_n$ are the moves that **I** made on the moves $0, \ldots, n$.*

7. *A Borel*-code (T, π) is a Borel*-code for $X \subseteq \mathbf{B}$ if for all $\eta \in \mathbf{B}$, $\eta \in X$ if and only if **II** has a winning strategy in $GB^*(\eta, (T, \pi))$. We say that $X \subseteq \mathbf{B}$ is a Borel* set if it has a Borel*-code. We denote by Borel*$(\mathbf{B})$ the class of Borel* sets.*

Theorem 1.9. *Borel$(\mathbf{B})$ = Borel*$(\mathbf{B})$.*

Proof. Let us start by showing that *Borel*$(\mathbf{B}) \subseteq$ *Borel**$(\mathbf{B})$. We will prove this by showing that every open set is a *Borel** set and if $\{X_i\}_{i<\omega}$ is a countable collection of *Borel** sets, then $\bigcup_{i<\omega} X_i$ and $\bigcap_{i<\omega} X_i$ are *Borel** sets.

Suppose that X is an open set. Let $\{\xi_i\}_{i<\omega}$ be a collection of elements of $\omega^{<\omega}$ such that $X = \bigcup_{i<\omega} N_{\xi_i}$. Let $T = (\omega \times \omega)^{\leq 1}$ and π the fuction given by $\pi((0, (i, j))) = N_{\xi_j}$. It is clear that for every $\eta \in X$, **II** has a winning strategy in $GB^*(\eta, (T, \pi))$. Therefore (T, π) is a *Borel**-code for X.

Suppose that $\{X_i\}_{i<\omega}$ is a countable collection of *Borel** sets. Let (T_i, π_i) be a *Borel**-code of X_i. Let T be the set of all functions $f : n \to (\omega \times \omega)$, for some $n < \omega$, such that if $f(0) = (i, j)$, then there is $g \in T_i$, $g : n-1 \to (\omega \times \omega)$ with $dom(f) = dom(g) + 1$, and $f(m) = g(m-1)$, for all $0 < m < dom(f)$. For every leaf f of T if $f(0) = (i, j)$, then there is $g \in L(T_i)$ such that $f(m) = g(m-1)$, for all $0 < m < dom(f)$; define $\pi(f) = \pi_i(g)$.

Claim 1.10. *(T, π) is a Borel*-code of $\bigcap_{i<\omega} X_i$, and $\bigcap_{i<\omega} X_i$ is a Borel* set.*

Proof. Let $\eta \in \bigcap_{i<\omega} X_i$. Then for all $i < \omega$, there is a winning strategy W_i of **II** in $GB^*(\eta, (T_i, \pi_i))$. Define $W : \omega^{<\omega} \to \omega$ by $W(i_0) = 0$ and $W(i_0, \ldots, i_n) = W_{i_0}(i_1, \ldots, i_n)$ for all $0 < n < \omega$. It is easy to see that W is a winning strategy of **II** in $GB^*(\eta, (T, \pi))$.

Let $\eta \in \mathbf{B}$ be such that **II** has a winning strategy, W, in $GB^*(\eta, (T, \pi))$. Define $W_i : \omega^{<\omega} \to \omega$ by $W_i(i_0, \ldots, i_n) = W(i, i_0, \ldots, i_n)$. It is easy to see that W_i is a winning strategy of **II** in $GB^*(\eta, (T_i, \pi_i))$. Since this holds for all $i < \omega$, we conclude that $\eta \in X_i$, for all $i < \omega$. □

Let (T_i, π_i) be a *Borel**-code of X_i. Let T be the set of all functions $f : n \to (\omega \times \omega)$, for some $n < \omega$, such that if $f(0) = (i, j)$, then there is $g \in T_j$, $g : n-1 \to (\omega \times \omega)$ with $dom(f) = dom(g) + 1$ and $f(m) = g(m-1)$, for all $0 < m < dom(f)$. For every leaf f of T if $f(0) = (i, j)$, then there is $g \in L(T_j)$ such that $f(m) = g(m-1)$, for all $0 < m < dom(f)$; define $\pi(f) = \pi_j(g)$.

Claim 1.11. *(T, π) is a Borel*-code of $\bigcup_{i<\omega} X_i$, and $\bigcup_{i<\omega} X_i$ is a Borel* set.*

Proof. Let $\eta \in \bigcup_{i<\omega} X_i$. Then there is $j < \omega$, such that there is a winning strategy W_j of **II** in $GB^*(\eta, (T_j, \pi_j))$. Define $W : \omega^{<\omega} \to \omega$ by $W(i_0) = j$ and $W(i_0, \ldots, i_n) = W_j(i_1, \ldots, i_n)$ for all $0 < n < \omega$. It is easy to see that W is a winning strategy of **II** in $GB^*(\eta, (T, \pi))$.

Let $\eta \in \mathbf{B}$ be such that **II** has a winning strategy, W, in $GB^*(\eta, (T, \pi))$. Define $W' : \omega^{<\omega} \to \omega$ by $W'(i_1, \ldots, i_n) = W(0, \ldots, i_n)$. It is easy to see that W' is a winning strategy of **II** in $GB^*(\eta, (T_{W(0)}, \pi_{W(0)}))$. Therefore $\eta \in X_{W(0)}$. □

To show that *Borel**$(\mathbf{B}) \subseteq$ *Borel*$(\mathbf{B})$ we will define the rank of an ω-tree and the rank of the elements of an ω-tree.

Given an ω-tree T, we define the rank function, rk, as follows:

- If $\eta \in L(T)$, then $rk(\eta) = 0$.
- If $\eta \notin L(T)$, then $rk(\eta) = \bigcup\{rk(f) + 1 \mid f^- = \eta\}$.

The rank of a tree T is defined by $rk(T) = rk(\emptyset)$.

Exercise 1.3. *1. Show that the rank of an ω-tree is smaller than ω_1.*

2. Find an ω-tree with infinite rank.

Let X be a *Borel** set, and (T, π) a *Borel**-code of X. We will prove by induction on $rk(T)$ that X is a Borel set.

Case $rk(T) = 0$. It is clear that $T = \{\emptyset\}$ and $X = \pi(\emptyset)$, therefore X is a Borel set.

Suppose $rk(T) = \alpha$ and if Y is *Borel** set with *Borel**-code (T', π') with $rk(T) < \alpha$, then Y is a Borel set.

Let T_{ij} be the set of all functions $f : n \to \omega$ such that there is a function $g \in T$ with $g(0) = (i, j)$, $dom(g) = dom(f) + 1$ and $f(m) = g(m + 1)$ for all $m \in dom(f)$. Define π_{ij} by $\pi_{ij}(f) = \pi(g)$, where $g \in T$ is such that $g(0) = (i, j)$, $dom(g) = dom(f) + 1$ and $f(m) = g(m + 1)$ for all $m \in dom(f)$. Notice that for all $i, j < \omega$, $rk(T_{ij}) < \alpha$. By the induction hypothesis, for all $i, j < \omega$, (T_{ij}, π_{ij}) is a *Borel**-code of a Borel set. Denote by B_{ij} the Borel set with *Borel**-code (T_{ij}, π_{ij}).

Claim 1.12. $X = \bigcap_{i<\omega} \bigcup_{j<\omega} B_{ij}$

Proof. Let $\eta \in X$, then **II** has a winning strategy, W, in $GB^*(\eta, (T, \pi))$. Define $W_{iW(i)} : \omega^{<\omega} \to \omega$ by $W_{iW(i)}(i_0, \ldots, i_n) = W(i, i_0, \ldots, i_n)$, it is clear that $W{-}iW(i)$ is a winning strategy of **II** in $GB^*(\eta, (T_{iW(i)}, \pi_{iW(i)}))$, so $\eta \in B_{iW(i)}$. Therefore, for all $i < \omega$ there is $j < \omega$ such that $\eta \in B_{ij}$, we conclude that $\eta \in \bigcap_{i<\omega} \bigcup_{j<\omega} B_{ij}$.

Let $\eta \in \bigcap_{i<\omega} \bigcup_{j<\omega} B_{ij}$. Then for all $i < \omega$ there is $j < \omega$ such that $\eta \in B_{ij}$, denote by $h(i)$ this j. So there is $W_{ih(i)}$ a winning strategy of **II** in $GB^*(\eta, (T_{ih(i)}, \pi_{ih(i)}))$. Define $W : \omega^{<\omega} \to \omega$ by $W(i_0) = h(i_0)$ and $W(i_0, \ldots, i_n) = W_{h(i_0)}(i_1, \ldots, i_n)$. It is clear that W is a winning strategy of **II** in $GB^*(\eta, (T_{iW(i)}, \pi_{iW(i)}))$ and $\eta \in X$. □

□

At the beginning the *Borel**-codes look very artificial and complicated, but this codes will be very helpful in the future. In order to give a better understanding of the motivation behind the *Borel**-codes we will define the *Borel***-codes. This codes use intersections and unions as part of the coding of sets, this gives a better understanding on what is going on in the coding.

Definition 1.13. *1. A pair (T, π) is a Borel***-code *if $T \subseteq \omega^{<\omega}$ is an ω-tree and π is a function with domain T such that if $f \in T$ is a leaf, then $\pi(f)$ is an open set, and in case f is a node, $\pi(f) = \cap$ if $\mid dom(f) \mid$ is an even number and $\pi(f) = \cup$ if $\mid dom(f) \mid$ is an odd number.*

*2. For an element $\eta \in \mathbf{B}$ and a Borel**-code (T, π), the game $B^*(\eta, (T, \pi))$ is played as follows. There are two players, **I** and **II**. The game starts from the root of T. At each move, if the game is at node $f \in T$ and $\pi(f) = \cap$, then **I** chooses an immediate successor g of f and the game continues from this g. If $\pi(f) = \cup$, then **II** makes the choice. Finally, if $\pi(f)$ is an open set, then the game ends, and **II** wins if and only if $\eta \in \pi(x)$.*

3. A set $X \subseteq \omega^\omega$ is a Borel**-set *if there is a Borel**-code (T, π) such that for all $\eta \in \omega^\omega$, $\eta \in X$ if and only if **II** has a winning strategy in the game $B^*(\eta, (T, \pi))$. We denote by Borel$^{**}(\mathbf{B})$ the set of Borel** sets.*

Exercise 1.4. *Borel*$^*(\mathbf{B}) =$ *Borel*$^{**}(\mathbf{B})$.

Notice that the rank was defined for ω-trees in general. For every *Borel*** set, X, as the least ordinal α such that there is a *Borel***-code of X.

Exercise 1.5. *What is the relation between the rank of a Borel** set and the Borel hierarchy?*

Definition 1.14.
- *$X \subseteq \mathbf{B}$ is $\Sigma^1_1(\mathbf{B})$ if there is $Y \subseteq \mathbf{B} \times \mathbf{B}$ a Borel set such that $pr(Y) = X$.*
- *$X \subseteq \mathbf{B}$ is $\Pi^1_1(\mathbf{B})$ if $\mathbf{B} \backslash X$ is $\Sigma^1_1(\mathbf{B})$.*
- *$X \subseteq \mathbf{B}$ is $\Delta^1_1(\mathbf{B})$ if it is $\Sigma^1_1(\mathbf{B})$ and $\Pi^1_1(\mathbf{B})$.*

Lemma 1.15. *The following are equivalent:*

- *X is $\Sigma^1_1(\mathbf{B})$.*
- *$X = pr(Y)$ for some closed $y \subseteq \mathbf{B} \times \mathbf{B}$.*

Lemma 1.16. *If $X \subseteq \mathbf{B}$ is Borel, then X is $\Delta_1^1(\mathbf{B})$.*

Proof. Let $X \subseteq \mathbf{B}$ be a Borel set and (T, π) a $Borel^*$-code for X. Let $h : \omega^{<\omega} \to \omega$ be one-to-on and onto. For all $f \in \omega^\omega$ define $W_f : \omega^{<\omega} \to \omega$ by $W_f(i_0, \ldots, i_n) = f(h(i_0, \ldots, i_n))$. Let P be the set of all the tuples $(\eta, f) \in \omega^\omega \times \omega^\omega$ such that W_f is a winning strategy for **II** in the game $GB^*(\eta, (T, \pi))$. It is clear that $pr(P) = X$.

Claim 1.17. *P is closed*

Proof. Let $(\eta, f) \notin P$ then there are $n < \omega$ and $\{j_0, \ldots, j_n\}$ such that if **I** choose j_m in the m-move and **II** choose $W_f(j_0 \ldots, j_m)$ in the m-move, then after n moves the game stops in a leaf g and $\eta \notin \pi(g)$. Therefore, there is $r < \omega$, such that $N_{\eta \restriction r} \cap \pi(g) = \emptyset$, so $(N_{\eta \restriction r} \times N_{f \restriction m}) \cap P = \emptyset$. □

We conclude that X is $\Sigma_1^1(\mathbf{B})$ and since $Borel(\mathbf{B})$ is closed under complements, we conclude that $\mathbf{B} \backslash X$ is Borel, therefore it is $\Sigma_1^1(\mathbf{B})$. We conclude that X is $\Delta_1^1(\mathbf{B})$. □

Exercise 1.6. *Prove the claims of the following proof.*

Theorem 1.18 (Separation)**.** *If $X, Y \subseteq \mathbf{B}$ are $\Sigma_1^1(\mathbf{B})$ disjoint sets, then there is a Borel set $Z \subseteq \mathbf{B}$ that satisfies $X \subseteq Z \subseteq \mathbf{B} \backslash Y$.*

Proof. Choose $X^*, Y^* \subseteq \mathbf{B} \times \mathbf{B}$ such that $pr(X^*) = X$ and $pr(Y^*) = Y$. For all $\eta \in \mathbf{B}$, let X_η be the set of all $\xi \in \omega^\omega$ that satisfy the following: If $dom(\xi) = n$, then there are $\eta'\xi' \in \mathbf{B}$, $(\eta', \xi') \in X^*$, and $\eta' \restriction n = \eta \restriction n$ and $\xi \subseteq \xi'$. Define Y_η in the same way. We denote by $X_{\eta \restriction n}$ the set of functions $\xi \in \omega^n$ such that there is $\eta' \in \mathbf{B}$, and $\xi \in X_{\xi'}$ and $\eta \restriction n \subseteq \eta'$. It is clear that $X_\eta = \bigcup_{n<\omega} X_{\eta \restriction n}$.

Given two trees $T, T' \subseteq \omega^{<\omega}$, we say that $T \leq T'$ if there is a function $f : T \to T'$ that satisfies the following: for all $\eta, \xi \in T$, if $\eta \subsetneq \xi$, then $f(\eta) \subsetneq f(\xi)$. Let Z be the set of $\eta \in \mathbf{B}$ that satisfy $Y_\eta \leq X_\eta$.

Claim 1.19.

- *If $\eta \in X$, then $Y_\eta \leq X_\eta$.*
- *If $Y_\eta \leq X_\eta$, then $\eta \notin Y$.*
- *$X \subseteq Z \subseteq \mathbf{B} \backslash Y$.*

for all $T, T' \subseteq \omega^{<\omega}$ we define the game $GC(T, T')$ as follows: in the n-th movement, **I** chooses $t_n \in T$ such that $t_m \subseteq t_n$ holds for all $m < n$, and **II** chooses $t'_n \in T'$ such that $t'_m \subseteq t'_n$ holds for all $m < n$. The game ends when a player cannot make a choice, the player that cannot make a choice looses.

Claim 1.20. *$T \leq T'$ si y solo si* **II** *has a winning strategy for the game $GC(T, T')$.*

Let T be the set of all functions with finite domain, $f : n \to \bigcup_{m<\omega} (\omega^m)^3$ such that for all $i < n$ the following holds:

- $f(i) \in (\omega^i)^3$.
- If $j + 1 < n$ and $f(j) = (\xi_k)_{k<3}$, then $\xi_1 \in X_{\xi_0}$ and $\xi_2 \in X_{\xi_0}$.
- If $j < l < n$, $f(j) = (\xi_k)_{k<3}$, and $f(l) = (\xi'_k)_{k<3}$, then for all $k < 3$, $\xi_k \subseteq \xi'_k$.

Define π with domain $L(T)$ as $\pi(f) = N_{\xi_0}$ if $dom(f) = n + 1$, $f(n) = (\xi_k)_{k<3}$, and $\xi_2 \notin Y_{\xi_0}$. And $\pi(f) = \emptyset$ in other case.

Claim 1.21. *There is a $Borel^*$-code (T', π') such that there is a tree isomorphism $h : T' \to T$ that satisfies $\pi'(f) = \pi(h(f))$.*

Claim 1.22. **II** *has a winning strategy in $GB^*(\eta, (T', \pi'))$ if and only if $GC(Y_\eta, X_\eta)$.*

□

The following is a standard way to code structures with domain ω with elements of 2^ω. Fix a countable relational vocabulary $\mathcal{L} = \{P_n \mid n < \omega\}$.

Definition 1.23. *Fix a bijection $\pi \colon \omega^{<\omega} \to \omega$. For every $\eta \in 2^\omega$ define the $\mathcal{L}$-structure $\mathcal{A}_\eta$ with domain ω as follows: For every relation P_m with arity n, every tuple $(a_1, a_2, \ldots, a_n)$ in ω^n satisfies*

$$(a_1, a_2, \ldots, a_n) \in P_m^{\mathcal{A}_\eta} \iff \eta(\pi(m, a_1, a_2, \ldots, a_n)) = 1.$$

Definition 1.24 (The isomorphism relation)**.** *Assume T is a complete first order theory in a countable vocabulary. We define $\cong_T^\omega$ as the relation*

$$\{(\eta, \xi) \in 2^\omega \times 2^\omega \mid (\mathcal{A}_\eta \models T, \mathcal{A}_\xi \models T, \mathcal{A}_\eta \cong \mathcal{A}_\xi) \text{ or } (\mathcal{A}_\eta \not\models T, \mathcal{A}_\xi \not\models T)\}.$$

A function $f\colon 2^\omega \to 2^\omega$ is Borel, if for every open set $A \subseteq 2^\omega$ the inverse image $f^{-1}[A]$ is a Borel subset of 2^ω. Let E_1 and E_2 be equivalence relations on 2^ω. We say that E_1 is Borel reducible to E_2, if there is a Borel function $f\colon 2^\omega \to 2^\omega$ that satisfies $(x,y) \in E_1 \Leftrightarrow (f(x), f(y)) \in E_2$, we denote it by $E_1 \leq_B E_2$.

Exercise 1.7. *A function f is Borel if and only if for all Borel set X, $f^{-1}[X]$ is Borel.*

Example 1.1. *Let T_1 be the theory of the order of the rational numbers, $\cong^\omega_{T_1}$ has only two equivalent classes. Let T_2 be the theory of a vector space over the field of rational numbers. $\cong^\omega_{T_1} \leq_B \cong^\omega_{T_2}$.*

This can be use to compare the complexity of two theories, from Example 1.1 we conclude that T_1 is less complex than T_2, in the Borel reducibility sense.

Question 1.25. *Is there an equivalence relation E on 2^ω such that for every complete first order theory in a countable vocabulary T, either $E \not\leq_B \cong^\omega_{T_1}$ or $\cong^\omega_{T_1} \not\leq_B E$.*

Let T be a complete countable theory, we will denote by $I(\lambda, T)$ the amount of non-isomorphic models of T of size λ. The following is the main theorem of [19].

Theorem 1.26 (The Main Gap Theorem, [19])**.** *Let T be a complete countable theory.*

- *If T is not superstable, or deep, or with DOP or OTOP then for every uncountable cardinal λ, $I(\lambda, T) = 2^\lambda$.*
- *If T is shallow superstable without DOP and without OTOP, then for every $\alpha > 0$, $I(\aleph_\alpha, T) \leq \beth_{\omega_1}(|\alpha|)$.*

Let T be a complete countable theory, we say that T is a classifiable theory if T is superstable without DOP and without OTOP. T_1 in Example 1.1 is not classifiable and T_2 is classifiable. The Main Gap Theorem tells us that classifiable theories are less complex than non-classifiable ones, in the stability sense.

2 Generalized Baire spaces

Generalized descriptive set theory is the generalization of descriptive set theory to uncountable cardinals. For a background on classical descriptive set theory see [11] or [12]. During this notes, κ will be an uncountable cardinal that satisfies $\kappa^{<\kappa} = \kappa$, unless otherwise is stated.

The aim of this first section is to introduce the notions of κ-Borel class, $\Delta^1_1(\kappa)$ class, κ-Borel* class, and show the relation between these classes.

Definition 2.1 (The Generalized Baire space $\mathbf{B}(\kappa)$)**.** *Let κ be an uncountable cardinal. The generalized Baire space is the set κ^κ endowed with the following topology. For every $\eta \in \kappa^{<\kappa}$, define the following basic open set*

$$N_\eta = \{f \in \kappa^\kappa \mid \eta \subseteq f\}$$

the open sets are of the form $\bigcup X$ where X is a collection of basic open sets.

Definition 2.2 (The Generalized Cantor space $\mathbf{C}(\kappa)$)**.** *Let κ be an uncountable cardinal. The generalized Cantor space is the set 2^κ endowed with the following topology. For every $\eta \in 2^{<\kappa}$, define the following basic open set*

$$N_\eta = \{f \in 2^\kappa \mid \eta \subseteq f\}$$

the open sets are of the form $\bigcup X$ where X is a collection of basic open sets.

Definition 2.3 (κ-Borel class)**.** *Let $S \in \{\mathbf{B}(\kappa), \mathbf{C}(\kappa)\}$. The class κ-Borel(S) of all κ-Borel sets in S is the least collection of subsets of S which contains all open sets and is closed under complements, unions and intersections both of length at most κ.*

Definition 2.4. *Let $S \in \{\mathbf{B}(\kappa), \mathbf{C}(\kappa)\}$.*

- *$X \subset S$ is a $\Sigma^1_1(\kappa)$ set if there is a set $Y \subset S \times S$ a closed set such that $pr(Y) = \{x \in S \mid \exists y \in S\ (x,y) \in Y\} = X$.*
- *$X \subset S$ is a $\Pi^1_1(\kappa)$ set if $S \backslash X$ is a $\Sigma^1_1(\kappa)$ set.*
- *$X \subset S$ is a $\Delta^1_1(\kappa)$ set if X is a $\Sigma^1_1(\kappa)$ set and a $\Pi^1_1(\kappa)$ set.*

Definition 2.5 (κ-*Borel**-set in $\mathbf{B}(\kappa), \mathbf{C}(\kappa)$)**.** *Let $S \in \{2^\kappa, \kappa^\kappa\}$.*

1. *A subset $T \subset \kappa^{<\kappa}$ is a tree if for all $f \in T$ with $\alpha = dom(f) > 0$ and for all $\beta < \alpha$, $f \restriction \beta \in T$ and $f \restriction \beta < f$.*

2. *A tree T is a κ^+,λ-tree if does not contain chains of length λ and its cardinality is less than κ^+. It is closed if every chain has a unique supremum in T.*

3. *A pair (T,h) is a κ-Borel*-code if T is a closed κ^+,λ-tree, $\lambda\leq\kappa$, and h is a function with domain T such that if $x\in T$ is a leaf, then $h(x)$ is a basic open set and otherwise $h(x)\in\{\cup,\cap\}$.*

4. *For an element $\eta\in S$ and a κ-Borel*-code (T,h), the κ-Borel*-game $B^*(T,h,\eta)$ is played as follows. There are two players, **I** and **II**. The game starts from the root of T. At each move, if the game is at node $x\in T$ and $h(x)=\cap$, then **I** chooses an immediate successor y of x and the game continues from this y. If $h(x)=\cup$, then **II** makes the choice. At limits the game continues from the (unique) supremum of the previous moves. Finally, if $h(x)$ is a basic open set, then the game ends, and **II** wins if and only if $\eta\in h(x)$.*

5. *A set $X\subseteq S$ is a κ-Borel*-set if there is a κ-Borel*-code (T,h) such that for all $\eta\in S$, $\eta\in X$ if and only if **II** has a winning strategy in the game $B^*(T,h,\eta)$.*

We will write $\mathbf{II}\uparrow B^*(T,h,\eta)$ when **II** has a winning strategy in the game $B^*(T,h,\eta)$.

Example 2.1. *Let $\mu<\kappa$ be a regular cardinal, we say that $X\subseteq\kappa$ is a μ-club if X is an unbounded set and it is closed under μ-limits.*

Let $\mu<\kappa$ be a regular cardinal. For all $\eta,\xi\in 2^\kappa$ we say that η and ξ are $=^2_\mu$ equivalent if the set $\{\alpha<\kappa\mid\eta(\alpha)=\xi(\alpha)\}$ contains a μ-club.

The relation $=^2_\omega$ is a κ-Borel* set. Let us define the following κ-Borel*-code (T,h):

- *$T=\{f\in\kappa^{<\omega+2}\mid f$ is strictly incresing$\}$.*
- *For f not a leave, $h(f)=\cup$ if $dom(f)$ is even and $h(f)=\cap$ if $dom(f)$ is odd.*
- *To define $h(f)$ for a leave f, first define the set $L(g)=\{f\in\kappa^{\omega+1}\mid g\subseteq f\}$ for all $g\in T$ with domain ω, and $\gamma_g=sup_{n<\omega}(g(n))$. Let $h\restriction L(g)$ be a bijection between $L(g)$ and the set $\{N_p\times N_q\mid p,q\in\kappa^{\gamma_g+1},p(\gamma_g)=q(\gamma_g)\}$.*

*Let us show that (T,h) codes $=^2_\omega$. Suppose $\eta=^2_\omega\xi$, so there is an ω-club C such that $\forall\alpha\in C$ $\eta(\alpha)=\xi(\alpha)$. The following is a winning strategy for **II** in the game $B^*(T,h,(\eta,\xi))$. For every even $n<\omega$, if the game is at f with $dom(f)=n$, **II** chooses an immediate successor f' of f, such that $f\subset f'$ and $f'(n)\in C$. Since C is closed under ω limits, after ω moves the game continues at $g\in\kappa^\omega$ strictly increasing with $\gamma=sup_{n<\omega}(g(n))\in C$. So there is G an immediate successor of g, such that $h(G)=N_{\eta\restriction\gamma}\times N_{\xi\restriction\gamma}$. Finally if **II** chooses G in the ω move, then **II** wins.*

For the other direction, suppose $\eta\neq^2_\omega\xi$, so there is $A\subset S^\kappa_\omega$ stationary (S^κ_ω is the set of ω-cofinal ordinals below κ) such that for all $\alpha\in S$, $\eta(\alpha)\neq\xi(\alpha)$.

*We will show that for every σ strategy of **II**, σ is not a winning strategy. Let σ be an strategy for **II**, this mean that σ is a function from $\kappa^{<\omega+1}\to\kappa$. Notice that if **II** follows σ as a strategy, then when the game is at f, $dom(f)=n$ even, **II** chooses f' such that $f\subset f'$ and $f'(n)=\sigma((f(0),f(1),\dots,f(n-1)))$. Let C be the set of closed points of σ, $C=\{\alpha<\kappa\mid\sigma(\alpha^{<\omega})\subseteq\alpha\}$, C is unbounded and closed under ω-limits. Therefore $C\cap A\neq\emptyset$. Let γ be the least element of $C\cap A$ that is an ω-limit of elements of C, and let $\{\gamma_n\}_{n<\omega}$ be a sequence of elements of C cofinal to γ. The following is a winning strategy for **I** in the game $B^*(T,h,(\eta,\xi))$, if **II** uses σ as an strategy.*

*When the game is at f with $dom(f)=n$, n odd, then **I** chooses an immediate successor f' of f, such that $f\subset f'$ and $f'(n)$ is the least element of $\{\gamma_n\}_{n<\omega}$ that is bigger than $f(n-1)$. This element always exists because $\{\gamma_n\}_{n<\omega}$ is cofinal to γ and $\gamma\in C$, γ is a closed point of σ. Since **I** is following σ as a strategy and γ is a closed point of σ, after ω moves the game continues at $g\in\kappa^\omega$ strictly increasing with $\gamma=sup_{n<\omega}(g(n))\in C\cap A$. Since $\eta(\gamma)\neq\xi(\gamma)$, there is no G immediate successor of g, such that $(\eta,\xi)\in h(G)$. So it does not matter what **II** chooses in the ω move, **I** will win.*

The previous definitions are the generalization of the notions of Borel, Δ^1_1, and Borel* from descriptive set theory, the spaces ω^ω and 2^ω. A classical result in descriptive set theory states that the Borel class, the Δ^1_1 class, and the Borel* class are the same. This doesn't hold in generalized descriptive set theory as we will see.

Theorem 2.6 ([2], Thm 17)**.** *κ-Borel$\subseteq$ κ-Borel**

Proof. Let us prove something even stronger. X is a κ-Borel set if and only if there is a κ-Borel*-code (T,h) such that (T,h) codes X and T is a κ^+,ω-tree.

Let us define the sets $(B_i)_{i\leq\kappa^+}$ by:

- $B_0=\{N_p\mid p\in 2^{<\kappa}\}$, the set of basic open sets.

- If $\alpha = \beta + n$ for n an odd natural number and β a limit ordinal or 0, then $B_\alpha = B_{\beta+n-1} \cup \{\bigcap \mathcal{B} \mid \mathcal{B} \subseteq B_{\beta+n-1}, \mid \mathcal{B} \mid \leq \kappa\}$.
- If $\alpha = \beta + n$ for n an even positive natural number and β a limit ordinal or 0, then $B_\alpha = B_{\beta+n-1} \cup \{\bigcup \mathcal{B} \mid \mathcal{B} \subseteq B_{\beta+n-1}, \mid \mathcal{B} \mid \leq \kappa\}$.
- If α is a limit ordinal, then $B_\alpha = \bigcup_{\beta<\alpha} B_\beta$.

We will show by induction over α that for every $X \in B_\alpha$, there is a κ-Borel*-code (T, h) such that (T, h) codes X and T is a κ^+, ω-tree.

For $\alpha = 0$. If $X \in B_0$, then $T = \{\emptyset\}$ and $h(\emptyset) = X$ is a κ-Borel*-code that codes X.

Suppose $\alpha = \beta + n$ for n an even natural number and β a limit ordinal or 0 is such that for all $X \in B_\alpha$, there is a κ-Borel*-code (T, h) such that (T, h) codes X and T is a κ^+, ω-tree. Suppose $X \in B_{\alpha+n+1}$, so either $X \in B_\alpha + n$ or $X = \bigcap \mathcal{B}$ for some $\mathcal{B} \subseteq B_{\beta+n}$ with $\mid \mathcal{B} \mid = \gamma \leq \kappa$. Let $\mathcal{B} = \{X_i\}_{i<\gamma}$, by the induction hypothesis we know that there are κ-Borel*-code $\{(T_i, h_i)\}_{i<\gamma}$ such that (T_i, h_i) codes X_i and T_i is a κ^+, ω-tree, for all $i < \gamma$. Let $\mathcal{T} = \{r\} \cup \bigcup_{i<\gamma} T_i \times \{i\}$ be the tree ordered by $r < (x, j)$ for all $(x, j) \in \bigcup_{i<\gamma} T_i \times \{i\}$, and $(x, i) < (y, j)$ if and only if $i = j$ and $x < y$ in T_i. Let $T \subseteq \kappa^{<\omega}$ be a tree isomorphic to $\mathcal{T}$ and let $\mathcal{G} : T \to \mathcal{T}$ be a tree isomorphism. If $\mathcal{G}(x) \neq r$, then denote $\mathcal{G}(x)$ by $(\mathcal{G}_1(x), \mathcal{G}_2(x))$. Define h by $h(x) = \cap$ if $G(x) = r$, and $h(x) = h_{\mathcal{G}_2(x)}(\mathcal{G}_1(x))$.

Let us show that (T, h) codes X. Let $\eta \in X$, so $\eta \in X_i$ for all $i < \gamma$. If at the beginning **I** chooses x, then **II** follows the winning strategy from the game $B^*(T_{\mathcal{G}_2(x)}, h_{\mathcal{G}_2(x)}, \eta)$, choosing the element given by $\mathcal{G}^{-1}$. We conclude that **II** $\uparrow B^*(T, h, \eta)$. Let $\eta \notin X$, so there is $i < \gamma$ such that $\eta \notin X_i$, so **II** has no winning strategy for the game $B^*(T_i, h_i, \eta)$. Since at the beginning **I** can choose x such that $\mathcal{G}_2(x) = i$, **II** cannot have a winning strategy for the game $B^*(T, h, \eta)$. Otherwise **II** would have a winning strategy the game $B^*(T_i, h_i, \eta)$.

The case $\alpha = \beta + n$ for n an odd natural number and β a limit ordinal or 0 is similar, just make $h(x) = \cup$ if $G(x) = r$ when constructing (T, h).

Suppose α is a limit ordinal such that for all $\beta < \alpha$, for all $X \in B_\beta$, there is a κ-Borel*-code (T, h) such that (T, h) codes X and T is a κ^+, ω-tree. Let $X \in B_\alpha$, since $B_\alpha = \bigcup_{\beta<\alpha} B_\beta$ there is $\beta < \alpha$ such that $X \in B_\beta$. By the induction hypothesis, there is a κ-Borel*-code (T, h) such that (T, h) codes X and T is a κ^+, ω-tree. □

Theorem 2.7 ([2], Thm 17)**.** *1. κ-Borel* $\subseteq \Sigma^1_1(\kappa)$.*

2. κ-Borel$\subseteq \Sigma^1_1(\kappa)$.

3. κ-Borel$\subseteq \Delta^1_1(\kappa)$.

Proof. 1. Let X be a κ-Borel* set, there is a κ-Borel* code (T, h) such that X is coded by (T, h).

Since $\kappa^{<\kappa} = \kappa$, we can code the strategies $\sigma : T \to T$ by elements of κ^κ.

Claim 2.8. *The set $Y = \{(\eta, \xi) \mid \xi$ is a code of a winning strategy for* **II** *in $B^*(T, h, \eta)\}$ is closed.*

Proof. Let (η, ξ) be an element not in Y. So ξ is not a winning strategy for **II** in $B^*(T, h, \eta)\}$, there is $\alpha < \kappa$ such that for every $\zeta \in N_{\xi\restriction\alpha}$, ζ is not a winning strategy for **II** in $B^*(T, h, \eta)\}$. Otherwise T would have a branch of length κ. Because of the same reason, there is $\beta < \kappa$ such that for every $f \in N_{\eta\restriction\beta}$, $\zeta \in N_{\xi\restriction\alpha}$, ζ is not a winning strategy for **II** in $B^*(T, h, f)\}$. So $N_{\eta\restriction\beta} \times N_{\xi\restriction\alpha}$ is a subset of the complement of Y. □

Since $pr(Y) = X$, we are done.

2. It follows from Theorem 2.6 and (1).

3. It follows from (2) and the fact that κ-Borel sets are closed under complement.

□

The following theorem is the separation theorem and the proof can be found in [14].

Theorem 2.9 ([14], Corollary 34)**.** *Suppose A and B are disjoint $\Sigma^1_1(\kappa)$ sets. There are κ-Borel* sets C_0 and C_1 such that $A \subseteq C_0$, $B \subseteq C_1$, and C_0 and C_1 are duals.*

Theorem 2.10 ([2], Theorem 17)**.** *$\Delta^1_1(\kappa) \subseteq \kappa$-Borel**

Proof. Let A be a $\Delta^1_1(\kappa)$ set. Let $B = \mathbf{B}(\kappa)\backslash A$, by 2.9, there are κ-Borel* sets C_0 and C_1 such that $A \subseteq C_0$, $B \subseteq C_1$, and C_0 and C_1 are duals. Since C_0 and C_1 are duals, C_0 and C_1 are disjoint. So $A = C_0$, $B = C_1$. □

Corollary 2.11 ([14], Corollary 35)**.** *X is $\Delta_1^1(\kappa)$ if there is a κ-Borel*-code (T,h) that codes X and*

$$\mathbf{II} \uparrow B^*(T,h,\eta) \Leftrightarrow \mathbf{I} \not\uparrow B^*(T,h,\eta)$$

for all $\eta \in \kappa^\kappa$ the game is determined.

Exercise 2.1. *Prove the claims of the following proof.*

Theorem 2.12 ([2], Theorem 18)**.** *1. κ-Borel$\subsetneq \Delta_1^1(\kappa)$*

2. $\Delta_1^1(\kappa) \subsetneq \Sigma_1^1(\kappa)$

Proof. 1. Let $\xi \mapsto (T_\xi, h_\xi)$ be a continuous coding of the κ-Borel*-codes with T a $\kappa^+\omega$-tree, such that for all $\kappa^+\omega$-tree, T, and h, there is ξ such that $T_\xi, h_\xi = (T,h)$.

Claim 2.13. *The set $B = \{(\eta,\xi) \mid \eta$ is in the set coded by $(T_\xi, h_\xi)\}$ is $\Delta_1^1(\kappa)$ and is not κ-Borel, otherwise $D = \{\eta \mid (\eta,\eta) \notin B\}$ would be Borel.*

(Hint: use the set $C = \{(\eta,\xi,\sigma) \mid \sigma$ is a winning strategy for **II** *in $B^*(T_\xi,h_\xi,\eta)\}$).*

2.

Claim 2.14. *There is $A \subseteq 2^\kappa \times 2^\kappa$ such that if $B \subseteq 2^\kappa$ is a $\Sigma_1^1(\kappa)$ set, then there is $\eta \in 2^\kappa$ such that $B = \{\xi \mid (\xi,\eta) \in A\}$.*

(Hint: the construction used in the classical case works too).

The set $D = \{\eta \mid (\eta,\eta) \in A\}$ is $\Sigma_1^1(\kappa)$ but not $\Pi_1^1(\kappa)$.

□

From the previous results, we can see that

$$\kappa\text{-Borel} \subsetneq \Delta_1^1(\kappa) \subsetneq \Sigma_1^1(\kappa)$$

and

$$\Delta_1^1(\kappa) \subseteq \kappa\text{-Borel}^* \subseteq \Sigma_1^1(\kappa).$$

Therefore we are missing to determine whether one of the following holds:

- $\Delta_1^1(\kappa) \subsetneq \kappa\text{-Borel}^* \subsetneq \Sigma_1^1(\kappa)$;
- $\Delta_1^1(\kappa) \subsetneq \kappa\text{-Borel}^* = \Sigma_1^1(\kappa)$;
- $\Delta_1^1(\kappa) = \kappa\text{-Borel}^* \subsetneq \Sigma_1^1(\kappa)$.

As we will see, only case has not been answered.

Question 2.15. *Is the following consistent $\Delta_1^1(\kappa) = \kappa$-Borel* $\subsetneq \Sigma_1^1(\kappa)$?*

3 Reflection of Π_2^1-sentences

In this session we will focus on proving the consistency ofκ-Borel* $= \Sigma_1^1(\kappa)$. This was initially proved by Friedman-Hyttinen-Weisnstein in [2] under the assumption $V = L$.

Theorem 3.1 ([2], Theorem 18)**.** *If $V = L$, then κ-Borel* $= \Sigma_1^1(\kappa)$.*

We will show another proof which shows that κ-Borel* $= \Sigma_1^1(\kappa)$ holds in L but it can also be forced.

A function $f\colon \kappa^\kappa \to \kappa^\kappa$ is κ-Borel, if for every open set $A \subseteq \kappa^\kappa$ the inverse image $f^{-1}[A]$ is a κ-Borel subset of X. If Q_1 and Q_2 are quasi-orders on $\mathbb{B}_1, \mathbb{B}_2 \in \{2^\kappa, \kappa^\kappa\}$, respectively, then we say that Q_1 is *Borel-reducible* to Q_2 if there exists a κ-Borel map $f\colon 2^\kappa \to 2^\kappa$ such that for all $\eta, \xi \in 2^\kappa$ we have $\eta Q_1 \xi \iff f(\eta) Q_2 f(\xi)$ and this is also denoted by $Q_1 \hookrightarrow_B Q_2$.

Fact 3.2. *Assume $f\colon 2^\kappa \to 2^\kappa$ is a κ-Borel function and $B \subset 2^\kappa$ is κ-Borel*. Then $f^{-1}[B]$ is κ-Borel*.*

Proof. Let (T_B, H_B) be a κ-Borel*-code for B. Define the κ-Borel*-code (T_A, H_A) by letting $T_B = T_A$ and $H_A(b) = f^{-1}[H_B(b)]$ for every branch b of T_B. Let A be the κ-Borel*-set coded by (T_A, H_A). Clearly, **II** $\uparrow B^*(T_B, H_B, \eta)$ if and only if **II** $\uparrow B^*(T_A, H_A, f^{-1}(\eta))$, so $f^{-1}[B] = A$. □

The idea: Find a κ-Borel* equivalence relation R such that for all $\Sigma_1^1(\kappa)$ equivalence, Q, $Q \hookrightarrow_B R$.

A quasi-order is Σ_1^1-*complete*, if it is $\Sigma_1^1(\kappa)$ and every $\Sigma_1^1(\kappa)$ quasi-order is Borel-reducible to it. We will find a Σ_1^1-complete R that is κ-Borel*. Before we prove the result, let us take a look to the weakly compact cardinal to understand the motivation behind the definition of the diamond principle $\mathrm{Dl}_S^*(\Pi_2^1)$.

Let us suppose κ is a Π_2^1-indescernible cardinal. We know that $Reg(\kappa)$ the set of regular cardinals below κ is stationary. Therefore, we can define the equivalence relation $=^\kappa_{Reg}$ by

$$\eta \ =^\kappa_{Reg}\ \xi \ \Leftrightarrow\ \{\alpha \in Reg \mid \eta(\alpha) \neq \xi(\alpha)\} \text{ is non-stationary}$$

Let us show that $=^\kappa_{Reg}$ is a Σ_1^1-complete equivalence relation.

Theorem 3.3 ([1] Thm 3.7)**.** *If κ is a Π_2^1–indescribable cardinal, then $=^\kappa_{Reg}$ is $\Sigma_1^1(\kappa)$–complete.*

Proof. Let E be a $\Sigma_1^1(\kappa)$ equivalence relation on κ^κ. Then there is a closed set C on $\kappa^\kappa \times \kappa^\kappa \times \kappa^\kappa$ such that $\eta\ E\ \xi$ if and only if there exists $\zeta \in \kappa^\kappa$ such that $(\eta, \xi, \zeta) \in C$. Let us define $U = \{(\eta \restriction \alpha, \xi \restriction \alpha, \zeta \restriction \alpha) \mid (\eta, \xi, \zeta) \in C \ \&\ \alpha < \kappa\}$, and for every $\gamma < \kappa$ define $C_\gamma = \{(\eta, \xi, \zeta) \in \gamma^\gamma \times \gamma^\gamma \times \gamma^\gamma \mid \forall \alpha < \gamma\ (\eta \restriction \alpha, \xi \restriction \alpha, \zeta \restriction \alpha) \in U\}$. Let $E_\gamma \subset \gamma^\gamma \times \gamma^\gamma$ be the relation defined by $(\eta, \xi) \in E_\gamma$ if and only if there exists $\zeta \in \gamma^\gamma$ such that $(\eta, \xi, \zeta) \in C_\gamma$. Since E is an equivalence relation, it follows that E_γ is reflexive and symmetric, but not necessary transitive. Let us define the reduction by

$$F(\eta)(\alpha) = \begin{cases} f_\alpha(\eta) \text{ if } E_\alpha \text{ is an equivalence relation and } \eta \restriction \alpha \in \alpha^\alpha \\ 0 \text{ otherwise.} \end{cases}$$

where $f_\alpha(\eta)$ is a code in $\kappa \backslash \{0\}$ for the E_α–equivalence class of η.

Let us prove that if $(\eta, \xi) \in E$, then $(F(\eta), F(\xi)) \in =^\kappa_{reg}$. Suppose $(\eta, \xi) \in E$. Then there is $\zeta \in \kappa^\kappa$ such that $(\eta, \xi, \zeta) \in C$ and for all $\alpha < \kappa$ we have that $(\eta \restriction \alpha, \xi \restriction \alpha, \zeta \restriction \alpha) \in U$. On the other hand, we know that there is a club D such that for all $\alpha \in D \cap Reg(\kappa)$, $\eta \restriction \alpha$, $\xi \restriction \alpha$, $\zeta \restriction \alpha \in \alpha^\alpha$. We conclude that for all $\alpha \in D \cap Reg(\kappa)$, if E_α is an equivalence relation, then $(\eta, \xi) \in E_\alpha$. Therefore, for all $\alpha \in D \cap Reg(\kappa)$, $F(\eta)(\alpha) = F(\xi)(\alpha)$, so $F(\eta) \ =^\kappa_{Reg}\ F(\xi)$. Let us prove that if $(\eta, \xi) \notin E$, then $F(\eta) \ \neq^\kappa_{Reg}\ F(\xi)$. Suppose η, $\xi \in \kappa^\kappa$ are such that $(\eta, \xi) \notin E$. We know that there is a club D such that for all $\alpha \in D \cap Reg(\kappa)$, $\eta \restriction \alpha$, $\xi \restriction \alpha \in \alpha^\alpha$.

Notice that because C is closed $(\eta, \xi) \notin E$ is equivalent to

$$\forall \zeta \in \kappa^\kappa\ (\exists \alpha < \kappa\ (\eta \restriction \alpha, \xi \restriction \alpha, \zeta \restriction \alpha) \notin U),$$

so the sentence $(\eta, \xi) \notin E$ is a Π_1^1 property of the structure $(V_\kappa, \in, U, \eta, \xi)$. On the other hand, the sentence $\forall \zeta_1, \zeta_2, \zeta_3 \in \kappa^\kappa[((\zeta_1, \zeta_2) \in E \wedge (\zeta_2, \zeta_3) \in E) \rightarrow (\zeta_1, \zeta_3) \in E]$ is equivalent to the sentence $\forall \zeta_1, \zeta_2, \zeta_3, \theta_1, \theta_2 \in \kappa^\kappa[\exists \theta_3 \in \kappa^\kappa(\psi_1 \vee \psi_2 \vee \psi_3)]$, where ψ_1, ψ_2 and ψ_3 are, respectively, the formulas $\exists \alpha_1 < \kappa\ (\zeta_1 \restriction \alpha_1, \zeta_2 \restriction \alpha_1, \theta_1 \restriction \alpha_1) \notin U$, $\exists \alpha_2 < \kappa\ (\zeta_2 \restriction \alpha_2, \zeta_3 \restriction \alpha_2, \theta_2 \restriction \alpha_2) \notin U$, and $\forall \alpha_3 < \kappa\ (\zeta_1 \restriction \alpha_3, \zeta_3 \restriction \alpha_3, \theta_3 \restriction \alpha_3) \in U$. Therefore, the sentence $\forall \zeta_1, \zeta_2, \zeta_3 \in \kappa^\kappa[((\zeta_1, \zeta_2) \in E \wedge (\zeta_2, \zeta_3) \in E) \rightarrow (\zeta_1, \zeta_3) \in E]$ is a Π_2^1 property of the structure $(V_\kappa, \in, U)$. It follows that the sentence

$$(D \text{ is unbounded in } \kappa) \wedge ((\eta, \xi) \notin E) \wedge (E \text{ is an equivalence relation}) \wedge (\kappa \text{ is regular})$$

is a Π_2^1 property of the structure $(V_\kappa, \in, U, \eta, \xi)$. By Π_2^1 reflection, we know that there are stationary many $\gamma \in Reg(\kappa)$ such that γ is a limit point of D, E_γ is an equivalence relation, and $(\eta \restriction \gamma, \xi \restriction \gamma) \notin E_\gamma$. We conclude that there are stationary many $\gamma \in Reg(\kappa)$ such that $f_\gamma(\eta) \neq f_\gamma(\xi)$, and hence $F(\eta) \ \neq^\kappa_{reg}\ F(\eta)$ □

As we can see from the previous theorem, Π_2^1 reflection implies that $=^\kappa_{Reg}$ is $\Sigma_1^1(\kappa)$-complete. Unfortunately $=^\kappa_{Reg}$ is not necessarily κ-Borel*. As we saw in the first session, $=^\kappa_\omega$ is a κ-Borel* equivalence relation. Therefore, if there is a Π_2^1 reflection notion on the set $\{\alpha < \kappa \mid cf(\alpha) = \omega\}$, then we conclude that κ-Borel* $= \Sigma_1^1(\kappa)$. Let us define a notion of reflection on ordinals of cofinality ω.

Exercise 3.1. *A set Q is $\Sigma_1^1(\kappa)$ if and only if there is a tree T on $\kappa^{<\kappa} \times \kappa^{<\kappa} \times \kappa^{<\kappa}$ such that $Q = pr([T])$, that is,*

$$(\eta, \xi) \in Q \iff \exists \zeta \in \kappa^\kappa\ \forall \tau < \kappa\ (\eta \restriction \tau, \xi \restriction \tau, \zeta \restriction \tau) \in T.$$

A Π_2^1-sentence ϕ is a formula of the form $\forall X \exists Y \varphi$ where φ is a first-order sentence over a relational language $\mathcal{L}$ as follows:

- $\mathcal{L}$ has a predicate symbol ϵ of arity 2;
- $\mathcal{L}$ has a predicate symbol $\mathbb{X}$ of arity $m(\mathbb{X})$;
- $\mathcal{L}$ has a predicate symbol $\mathbb{Y}$ of arity $m(\mathbb{Y})$;

- $\mathcal{L}$ has infinitely many predicate symbols $(\mathbb{A}_n)_{n\in\omega}$, each $\mathbb{A}_n$ is of arity $m(\mathbb{A}_n)$.

Definition 3.4. *For sets N and x, we say that N* sees *x iff N is transitive, p.r.-closed, and $x \cup \{x\} \subseteq N$.*

Suppose that a set N sees an ordinal α, and that $\phi = \forall X \exists Y \varphi$ is a Π^1_2-sentence, where φ is a first-order sentence in the above-mentioned language $\mathcal{L}$. For every sequence $(A_n)_{n\in\omega}$ such that, for all $n \in \omega$, $A_n \subseteq \alpha^{m(\mathbb{A}_n)}$, we write

$$\langle \alpha, \in, (A_n)_{n\in\omega}\rangle \models_N \phi$$

to express that the two hold:

1. $(A_n)_{n\in\omega} \in N$;
2. $\langle N, \in\rangle \models (\forall X \subseteq \alpha^{m(\mathbb{X})})(\exists Y \subseteq \alpha^{m(\mathbb{Y})})[\langle \alpha, \in, X, Y, (A_n)_{n\in\omega}\rangle \models \varphi]$, where:
   - $\in$ is the interpretation of ϵ;
   - X is the interpretation of $\mathbb{X}$;
   - Y is the interpretation of $\mathbb{Y}$, and
   - for all $n \in \omega$, A_n is the interpretation of $\mathbb{A}_n$.

We write α^+ for $|\alpha|^+$, and write $\langle \alpha, \in, (A_n)_{n\in\omega}\rangle \models \phi$ for

$$\langle \alpha, \in, (A_n)_{n\in\omega}\rangle \models_{H_{\alpha^+}} \phi.$$

Definition 3.5. *Let κ be a regular and uncountable cardinal, and $S \subseteq \kappa$ stationary.*
$\mathrm{Dl}^*_S(\Pi^1_2)$ *asserts the existence of a sequence $\vec{N} = \langle N_\alpha \mid \alpha \in S\rangle$ satisfying the following:*

1. *for every $\alpha \in S$, N_α is a set of cardinality $< \kappa$ that sees α;*
2. *for every $X \subseteq \kappa$, there exists a club $C \subseteq \kappa$ such that, for all $\alpha \in C \cap S$, $X \cap \alpha \in N_\alpha$;*
3. *whenever $\langle \kappa, \in, (A_n)_{n\in\omega}\rangle \models \phi$, with ϕ a Π^1_2-sentence, there are stationarily many $\alpha \in S$ such that $|N_\alpha| = |\alpha|$ and $\langle \alpha, \in, (A_n \cap (\alpha^{m(\mathbb{A}_n)}))_{n\in\omega}\rangle \models_{N_\alpha} \phi$.*

The principle $\mathrm{Dl}^*_S(\Pi^1_2)$ provide us the reflection principle that we need, let us show that there is a Σ^1_1-complete quasi-order of 2^κ.

Definition 3.6. *Given a stationary subset $S \subseteq \kappa$, we define a quasi-order $\subseteq^S$ over 2^κ by letting, for any two elements $\eta : 2 \to \kappa$ and $\xi : 2 \to \kappa$,*

$$\eta \subseteq^S \xi \text{ iff } \{\alpha \in S \mid \eta(\alpha) > \xi(\alpha)\} \text{ is nonstationary.}$$

Lemma 3.7 (Transversal lemma, [4], Prop 3.1)**.** *Suppose that $\langle N_\alpha \mid \alpha \in S\rangle$ is a $\mathrm{Dl}^*_S(\Pi^1_2)$-sequence, for a given stationary $S \subseteq \kappa$. For every Π^1_2-sentence ϕ, there exists a transversal $\langle \eta_\alpha \mid \alpha \in S\rangle \in \prod_{\alpha\in S} N_\alpha$ satisfying the following.*
For every $\eta \in \kappa^\kappa$, whenever $\langle \kappa, \in, (A_n)_{n\in\omega}\rangle \models \phi$, there are stationarily many $\alpha \in S$ such that

1. *$\eta_\alpha = \eta \restriction \alpha$, and*
2. *$\langle \alpha, \in, (A_n \cap (\alpha^{m(\mathbb{A}_n)}))_{n\in\omega}\rangle \models_{N_\alpha} \phi$.*

Exercise 3.2. *There is a first-order sentence ψ_{fnc} in the language with binary predicate symbols ϵ and $\mathbb{X}$ such that, for every ordinal α and every $X \subseteq \alpha \times \alpha$,*

$$(X \text{ is a function from } \alpha \text{ to } \alpha) \text{ iff } (\langle \alpha, \in, X\rangle \models \psi_{\mathrm{fnc}}).$$

Exercise 3.3. *Let α be an ordinal. Suppose that ϕ is a Σ^1_1-sentence involving a predicate symbol $\mathbb{A}$ and two binary predicate symbols $\mathbb{X}_0, \mathbb{X}_1$. Denote $R_\phi := \{(X_0, X_1) \mid \langle \alpha, \in, A, X_0, X_1\rangle \models \phi\}$. Then there are Π^1_2-sentences $\psi_{\mathrm{Reflexive}}$ and $\psi_{\mathrm{Transitive}}$ such that:*

1. *$(R_\phi \supseteq \{(\eta, \eta) \mid \eta \in \alpha^\alpha\})$ iff $(\langle \alpha, \in, A\rangle \models \psi_{\mathrm{Reflexive}})$;*
2. *(R_ϕ is transitive) iff $(\langle \alpha, \in, A\rangle \models \psi_{\mathrm{Transitive}})$.*

Definition 3.8. *Denote by* $\mathrm{Lev}_3(\kappa)$ *the set of level sequences in* $\kappa^{<\kappa}$ *of length* 3*:*

$$\mathrm{Lev}_3(\kappa) := \bigcup_{\tau<\kappa} \kappa^\tau \times \kappa^\tau \times \kappa^\tau.$$

Fix an injective enumeration $\{\ell_\delta \mid \delta < \kappa\}$ *of* $\mathrm{Lev}_3(\kappa)$*. For each* $\delta < \kappa$*, we denote* $\ell_\delta = (\ell^0_\delta, \ell^1_\delta, \ell^2_\delta)$*. We then encode each* $T \subseteq \mathrm{Lev}_3(\kappa)$ *as a subset of* κ^5 *via:*

$$T_\ell := \{(\delta, \beta, \ell^0_\delta(\beta), \ell^1_\delta(\beta), \ell^2_\delta(\beta)) \mid \delta < \kappa, \ell_\delta \in T, \beta \in \mathrm{dom}(\ell^0_\delta)\}.$$

Theorem 3.9 ([4], Thm 3.5)**.** *Suppose* $\mathrm{Dl}^*_S(\Pi^1_2)$ *holds for a given stationary* $S \subseteq \kappa$*.*
For every analytic quasi-order Q *over* κ^κ*,* $Q \hookrightarrow_B \subseteq^S$*.*

Proof. Let Q be an analytic quasi-order over κ^κ. Fix a tree T on $\kappa^{<\kappa} \times \kappa^{<\kappa} \times \kappa^{<\kappa}$ such that $Q = \mathrm{pr}([T])$, that is,

$$(\eta, \xi) \in Q \iff \exists \zeta \in \kappa^\kappa \ \forall \tau < \kappa \ (\eta \restriction \tau, \xi \restriction \tau, \zeta \restriction \tau) \in T.$$

We shall be working with a first-order language having a 5-ary predicate symbol $\mathbb{A}$ and binary predicate symbols $\mathbb{X}_0, \mathbb{X}_1, \mathbb{X}_2$ and ϵ. By Exercise 3.2, for each $i < 3$, let us fix a sentence ψ^i_{fnc} concerning the binary predicate symbol $\mathbb{X}_i$ instead of $\mathbb{X}$, so that

$$(X_i \in \kappa^\kappa) \text{ iff } (\langle \kappa, \in, A, X_0, X_1, X_2\rangle \models \psi^i_{\mathrm{fnc}}).$$

Define a sentence φ_Q to be the conjunction of four sentences: ψ^0_{fnc}, ψ^1_{fnc}, ψ^2_{fnc}, and

$$\forall \tau \exists \delta \forall \beta[\epsilon(\beta, \tau) \to \exists \gamma_0 \exists \gamma_1 \exists \gamma_2 (\mathbb{X}_0(\beta, \gamma_0) \wedge \mathbb{X}_1(\beta, \gamma_1) \wedge \mathbb{X}_2(\beta, \gamma_2) \wedge \mathbb{A}(\delta, \beta, \gamma_0, \gamma_1, \gamma_2))].$$

Set $A := T_\ell$ as in Definition 3.8. Evidently, for all $\eta, \xi, \zeta \in \mathcal{P}(\kappa \times \kappa)$, we get that

$$\langle \kappa, \in, A, \eta, \xi, \zeta\rangle \models \varphi_Q$$

iff the two hold:

1. $\eta, \xi, \zeta \in \kappa^\kappa$, and
2. for every $\tau < \kappa$, there exists $\delta < \kappa$, such that $\ell_\delta = (\eta \restriction \tau, \xi \restriction \tau, \zeta \restriction \tau)$ is in T.

Let $\phi_Q := \exists X_2(\varphi_Q)$. Then ϕ_Q is a Σ^1_1-sentence involving predicate symbols $\mathbb{A}, \mathbb{X}_0, \mathbb{X}_1$ and ϵ for which the induced binary relation

$$R_{\phi_Q} := \{(\eta, \xi) \in (\mathcal{P}(\kappa \times \kappa))^2 \mid \langle \kappa, \in, A, \eta, \xi\rangle \models \phi_Q\}$$

coincides with the quasi-order Q. Now, appeal to Exercise 3.3 with ϕ_Q to receive the corresponding Π^1_2-sentences $\psi_{\mathrm{Reflexive}}$ and $\psi_{\mathrm{Transitive}}$. Then, consider the following two Π^1_2-sentences:

- $\psi^0_Q := \psi_{\mathrm{Reflexive}} \wedge \psi_{\mathrm{Transitive}} \wedge \phi_Q$, and
- $\psi^1_Q := \psi_{\mathrm{Reflexive}} \wedge \psi_{\mathrm{Transitive}} \wedge \neg(\phi_Q)$.

Let $\vec{N} = \langle N_\alpha \mid \alpha \in S\rangle$ be a $\mathrm{Dl}^*_S(\Pi^1_2)$-sequence. Appeal to Lemma 3.7 with the Π^1_2-sentence ψ^1_Q to obtain a corresponding transversal $\langle \eta_\alpha \mid \alpha \in S\rangle \in \prod_{\alpha \in S} N_\alpha$. Note that we may assume that, for all $\alpha \in S$, $\eta_\alpha \in {}^\alpha\alpha$, as this does not harm the key feature of the chosen transversal.

For each $\eta \in \kappa^\kappa$, let

$$Z_\eta := \{\alpha \in S \mid A \cap \alpha^5 \text{ and } \eta \restriction \alpha \text{ are in } N_\alpha\}.$$

Claim 3.10. *Suppose* $\eta \in \kappa^\kappa$*. Then* $S \setminus Z_\eta$ *is nonstationary.*

Proof. Fix primitive-recursive bijections $c : \kappa^2 \leftrightarrow \kappa$ and $d : \kappa^5 \leftrightarrow \kappa$. Given $\eta \in \kappa^\kappa$, consider the club D_0 of all $\alpha < \kappa$ such that:

- $\eta[\alpha] \subseteq \alpha$;
- $c[\alpha \times \alpha] = \alpha$;
- $d[\alpha \times \alpha \times \alpha \times \alpha \times \alpha] = \alpha$.

Now, as $c[\eta]$ is a subset of κ, by the choice $\vec{N}$, we may find a club $D_1 \subseteq \kappa$ such that, for all $\alpha \in D_1 \cap S$, $c[\eta] \cap \alpha \in N_\alpha$. Likewise, we may find a club $D_2 \subseteq \kappa$ such that, for all $\alpha \in D_2 \cap S$, $d[A] \cap \alpha \in N_\alpha$.

For all $\alpha \in S \cap D_0 \cap D_1 \cap D_2$, we have

- $c[\eta \restriction \alpha] = c[\eta \cap (\alpha \times \alpha)] = c[\eta] \cap c[\alpha \times \alpha] = c[\eta] \cap \alpha \in N_\alpha$, and
- $d[A \cap \alpha^5] = d[A] \cap d[\alpha^5] = d[A] \cap \alpha \in N_\alpha$.

As N_α is p.r.-closed, it then follows that $\eta \restriction \alpha$ and $A \cap \alpha^5$ are in N_α. Thus, we have shown that $S \setminus Z_\eta$ is disjoint from the club $D_0 \cap D_1 \cap D_2$. □

For all $\eta \in \kappa^\kappa$ and $\alpha \in Z_\eta$, let:

$$\mathcal{P}_{\eta,\alpha} := \{p \in \alpha^\alpha \cap N_\alpha \mid \langle \alpha, \in, A \cap \alpha^5, p, \eta \restriction \alpha \rangle \models_{N_\alpha} \psi_Q^0\}.$$

Finally, define a function $f : \kappa^\kappa \to 2^\kappa$ by letting, for all $\eta \in \kappa^\kappa$ and $\alpha < \kappa$,

$$f(\eta)(\alpha) := \begin{cases} 1, & \text{if } \alpha \in Z_\eta \text{ and } \eta_\alpha \in \mathcal{P}_{\eta,\alpha}; \\ 0, & \text{otherwise.} \end{cases}$$

Exercise 3.4. *f is Borel.*

Claim 3.11. *Suppose $(\eta, \xi) \in Q$. Then $f(\eta) \subseteq^S f(\xi)$.*

Proof. As $(\eta, \xi) \in Q$, let us fix $\zeta \in \kappa^\kappa$ such that, for all $\tau < \kappa$, $(\eta \restriction \tau, \xi \restriction \tau, \zeta \restriction \tau) \in T$. Define a function $g : \kappa \to \kappa$ by letting, for all $\tau < \kappa$,

$$g(\tau) := \min\{\delta < \kappa \mid \ell_\delta = (\eta \restriction \tau, \xi \restriction \tau, \zeta \restriction \tau)\}.$$

As $(S \setminus Z_\eta)$, $(S \setminus Z_\xi)$ and $(S \setminus Z_\zeta)$ are nonstationary, let us fix a club $C \subseteq \kappa$ such that $C \cap S \subseteq Z_\eta \cap Z_\xi \cap Z_\zeta$. Consider the club $D := \{\alpha \in C \mid g[\alpha] \subseteq \alpha\}$. We shall show that, for every $\alpha \in D \cap S$, if $f(\eta)(\alpha) = 1$ then $f(\xi)(\alpha) = 1$.

Fix an arbitrary $\alpha \in D \cap S$ satisfying $f(\eta)(\alpha) = 1$. In effect, the following three conditions are satisfied:

1. $\langle \alpha, \in, A \cap \alpha^5 \rangle \models_{N_\alpha} \psi_{\text{Reflexive}}$,
2. $\langle \alpha, \in, A \cap \alpha^5 \rangle \models_{N_\alpha} \psi_{\text{Transitive}}$, and
3. $\langle \alpha, \in, A \cap \alpha^5, \eta_\alpha, \eta \restriction \alpha \rangle \models_{N_\alpha} \phi_Q$.

In addition, since α is a closure point of g, by definition of φ_Q, we have

$$\langle \alpha, \in, A \cap \alpha^5, \eta \restriction \alpha, \xi \restriction \alpha, \zeta \restriction \alpha \rangle \models \varphi_Q.$$

As $\alpha \in S$ and φ_Q is first-order,

$$\langle \alpha, \in, A \cap \alpha^5, \eta \restriction \alpha, \xi \restriction \alpha, \zeta \restriction \alpha \rangle \models_{N_\alpha} \varphi_Q,$$

so that, by definition of ϕ_Q,

$$\langle \alpha, \in, A \cap \alpha^5, \eta \restriction \alpha, \xi \restriction \alpha \rangle \models_{N_\alpha} \phi_Q.$$

By combining the preceding with clauses (2) and (3) above, we infer that the following holds, as well:

(4) $\langle \alpha, \in, A \cap \alpha^5, \eta_\alpha, \xi \restriction \alpha \rangle \models_{N_\alpha} \phi_Q$.

Altogether, $f(\xi)(\alpha) = 1$, as sought. □

Claim 3.12. *Suppose $(\eta, \xi) \in \kappa^\kappa \times \kappa^\kappa \setminus Q$. Then $f(\eta) \not\subseteq^S f(\xi)$.*

Proof. As $(S \setminus Z_\eta)$ and $(S \setminus Z_\xi)$ are nonstationary, let us fix a club $C \subseteq \kappa$ such that $C \cap S \subseteq Z_\eta \cap Z_\xi$. As Q is a quasi-order and $(\eta, \xi) \notin Q$, we have:

1. $\langle \kappa, \in, A \rangle \models \psi_{\text{Reflexive}}$,
2. $\langle \kappa, \in, A \rangle \models \psi_{\text{Transitive}}$, and
3. $\langle \kappa, \in, A, \eta, \xi \rangle \models \neg(\phi_Q)$.

so that, altogether,

$$\langle \kappa, \in, A, \eta, \xi \rangle \models \psi_Q^1.$$

Then, by the choice of the transversal $\langle \eta_\alpha \mid \alpha \in S \rangle$, there is a stationary subset $S' \subseteq S \cap C$ such that, for all $\alpha \in S'$:

1. $\langle \alpha, \in, A \cap \alpha^5 \rangle \models_{N_\alpha} \psi_{\text{Reflexive}}$,

2. $\langle \alpha, \in, A \cap \alpha^5 \rangle \models_{N_\alpha} \psi_{\text{Transitive}}$,

3. $\langle \alpha, \in, A \cap \alpha^5, \eta \restriction \alpha, \xi \restriction \alpha \rangle \models_{N_\alpha} \neg(\phi_Q)$, and

4. $\eta_\alpha = \eta \restriction \alpha$.

By Clauses (3') and (4'), we have that $\eta_\alpha \notin \mathcal{P}_{\xi,\alpha}$, so that $f(\xi)(\alpha) = 0$.
By Clauses (1'), (2') and (4'), we have that $\eta_\alpha \in \mathcal{P}_{\eta,\alpha}$, so that $f(\eta)(\alpha) = 1$.
Altogether, $\{\alpha \in S \mid f(\eta)(\alpha) > f(\xi)(\alpha)\}$ covers the stationary set S', so that $f(\eta) \not\subseteq^S f(\xi)$. □

This completes the proof of Theorem 3.9 □

Definition 3.13. *For a stationary $S \subseteq \kappa$, $\diamondsuit_S^{++}$ asserts the existence of a sequence $\langle K_\alpha \mid \alpha \in S \rangle$ satisfying the following:*

1. *for every infinite $\alpha \in S$, K_α is a set of size $|\alpha|$;*

2. *for every $X \subseteq \kappa$, there exists a club $C \subseteq \kappa$ such that, for all $\alpha \in C \cap S$, $C \cap \alpha, X \cap \alpha \in K_\alpha$;*

3. *the following set is stationary in $[H_{\kappa^+}]^{<\kappa}$:*

$$\{M \in [H_{\kappa^+}]^{<\kappa} \mid M \cap \kappa \in S \ \& \ \text{clps}(M, \in) = (K_{M \cap \kappa}, \in)\}.$$

Theorem 3.14 ([18], Prop 1.4)**.** *$\diamondsuit_S^{++}$ holds in L.*

Lemma 3.15 ([3], Thm 4.10)**.** *For every stationary $S \subseteq \kappa$, $\diamondsuit_S^{++}$ implies $\text{Dl}_S^*(\Pi_2^1)$.*

Definition 3.16. *Let $\mathbb{S}$ be the poset of all pairs $(k, \mathcal{B})$ with the following properties:*

1. *k is a function such that $dom(k) < \kappa$;*

2. *for each $\alpha \in dom(k), k(\alpha)$ is a transitive model of ZF^- of size $\leq \max\{\aleph_0, |\alpha|\}$, with $k \restriction \alpha \in k(\alpha)$;*

3. *$\mathcal{B}$ is a subset of $\mathcal{P}(\kappa)$ of size $\leq \text{dom}(k)$;*

$(k', \mathcal{B}') \leq (k, \mathcal{B})$ in $\mathbb{S}$ if the following holds:

(i) *$k' \supseteq k$, and $\mathcal{B}' \supseteq \mathcal{B}$;*

(ii) *for any $B \in \mathcal{B}$ and any $\alpha \in dom(k') \setminus dom(k)$, $B \cap \alpha \in k'(\alpha)$.*

Lemma 3.17 ([18], Prop 1.5)**.** *For every stationary $S \subseteq \kappa$, $V^{\mathbb{S}} \models \diamondsuit_S^{++}$.*

Let us denote by $\text{Dl}_\omega^*(\Pi_2^1)$ the principle $\text{Dl}_S^*(\Pi_2^1)$ when $S = \{\alpha < \kappa \mid cf(\alpha) = \omega\}$. Since $\diamondsuit_S^{++}$ holds in L, in L we have κ-Borel$^* = \Sigma_1^1(\kappa)$. Also there is a $< \kappa$-closed κ^+-cc forcing which forces κ-Borel$^* = \Sigma_1^1(\kappa)$.

Theorem 3.18 ([6], Corollary 3.2)**.** *It is consistent that $\Delta_1^1(\kappa) \subsetneq \kappa$-Borel$^* \subsetneq \Sigma_1^1(\kappa)$.*

As we have seen, the equivalence relations $=_\mu^\kappa$ and $=_\mu^2$ play a crucial role. It is clear that $\text{Dl}_\mu^*(\Pi_2^1)$ implies $=_\mu^\kappa \hookrightarrow_B =_\mu^2$.

Question 3.19. *Is $=_\mu^\kappa \hookrightarrow_B =_\mu^2$ a theorem of ZFC?*

4 A generalized Borel-reducibility counterpart of Shelah's main gap

Shelah's Main Gap Theorem states the following.

Theorem 4.1 ([19] Main Gap Theorem)**.** *For every T first order complete theory over a countable vocabulary. Let $I(T, \alpha)$ denote the number of non-isomorphic models of T with cardinality α. One of the following holds:*

1. *If T is shallow superstable without DOP and without OTOP, then $\forall \alpha > 0$ $I(T, \aleph_\alpha) \leq \beth_{\omega_1}(|\ \alpha\ |)$.*

2. *If T is not superstable, or superstable and deep or with DOP or with OTOP, then for every uncountable cardinal α, $I(T, \alpha) = 2^\alpha$.*

This gives us a notion of complexity, a theory is more complex if it has more models. Unfortunately, the main gap also tells us that with this notion of complexity a theory T is either too complex, for every uncountable cardinal α $I(T,\alpha)=2^\alpha$, or it is not so complex, i.e. $\forall\alpha>0$ $I(T,\aleph_\alpha)<\beth_{\omega_1}(|\ \alpha\ |)$. The aim of study the Main Gap in the generalized Borel reducibility hierarchy is to obtain a more refined complexity notion in which different theories have different complexities, and satisfies a counterpart of the Main Gap theorem:

If T_1 and T_2 are first order complete theories over a countable vocabulary such that T_1 satisfies the first item of the Main Gap and T_2 satisfies the second item of the Main Gap theorem, then T_1 is less complex than T_2.

With the notions explained in the previous session, we can define the desire complexity notion:

T_1 is as much as complex as T_2 if and only $\cong_{T_1}\hookrightarrow_B\cong_{T_2}$.

To study this notion of complexity for first order complete theories over countable vocabularies, we will divide the theories in two classes (as the Main Gap suggested), classifiable and non-classifiable theories. The only difference is that we will not require a theory to be shallow in order to be classifiable. Some authors require shallow for classifiable theories, we will see why in our case it make sense to not require it.

Definition 4.2.
- *A first order complete theory over a countable vocabulary, T, is classifiable if it is superstable without DOP and without OTOP.*
- *A first order complete theory over a countable vocabulary, T, is non-classifiable if it satisfies one of the following:*
  1. *T is stable unsuperstable;*
  2. *T is superstable with DOP;*
  3. *T is superstable with OTOP;*
  4. *T is unstable.*

Let us fix a bijection $\pi:\kappa^{<\omega}\to\kappa$.

Definition 4.3. *For every $\eta\in\kappa^\kappa$ define the structure $\mathcal{A}_\eta$ with domain κ as follows. For every tuple $(a_1,a_2,\dots,a_n)$ in κ^n*

$$(a_1,a_2,\dots,a_n)\in P_m^{\mathcal{A}_\eta}\Leftrightarrow \text{ the arity of } P_m \text{ is } n \text{ and } \eta(\pi(m,a_1,a_2,\dots,a_n))>0.$$

Definition 4.4. *For every $\eta\in 2^\kappa$ define the structure $\mathcal{A}_\eta$ with domain κ as follows. For every tuple $(a_1,a_2,\dots,a_n)$ in κ^n*

$$(a_1,a_2,\dots,a_n)\in P_m^{\mathcal{A}_\eta}\Leftrightarrow \text{ the arity of } P_m \text{ is } n \text{ and } \eta(\pi(m,a_1,a_2,\dots,a_n))=1.$$

Notice that the structure $\mathcal{A}_\eta\restriction\alpha$ is not necessary coded by the function $\eta\restriction\alpha$.

Exercise 4.1. *There is a club C_π such that for all $\alpha\in C_\pi$, $\mathcal{A}_\eta\restriction\alpha=\mathcal{A}_{\eta\restriction\alpha}$*

With the structures coded by the elements of 2^κ and κ^κ, it is easy to define the isomorphism relation of structures of size κ in both spaces.

Definition 4.5 (The isomorphism relation)**.** *Assume T is a complete first order theory in a countable vocabulary. We define $\cong_T^\kappa$ as the relation*

$$\{(\eta,\xi)\in\kappa^\kappa\times\kappa^\kappa\mid(\mathcal{A}_\eta\models T,\mathcal{A}_\xi\models T,\mathcal{A}_\eta\cong\mathcal{A}_\xi) \text{ or } (\mathcal{A}_\eta\not\models T,\mathcal{A}_\xi\not\models T)\}.$$

Definition 4.6. *Assume T is a complete first order theory in a countable vocabulary. We define $\cong_T^2$ as the relation*

$$\{(\eta,\xi)\in 2^\kappa\times 2^\kappa\mid(\mathcal{A}_\eta\models T,\mathcal{A}_\xi\models T,\mathcal{A}_\eta\cong\mathcal{A}_\xi) \text{ or } (\mathcal{A}_\eta\not\models T,\mathcal{A}_\xi\not\models T)\}.$$

Notice that $\cong_T^\kappa\hookrightarrow_c\cong_T^2$ holds for every theory T. From now on let us denote by $\cong_t$ both notions $\cong_T^\kappa$ and $\cong_T^2$.

Let us start with the case of classifiable theories. The following is the usual Ehrenfeucht-Fraïssé game but coded in a particular way for our purposes.

Definition 4.7. *(Ehrenfeucht-Fraïssé game) Fix $\{X_\gamma\}_{\gamma<\kappa}$ an enumeration of the elements of $\mathcal{P}_\kappa(\kappa)$ and $\{f_\gamma\}_{\gamma<\kappa}$ an enumeration of all the functions with domain in $\mathcal{P}_\kappa(\kappa)$ and range in $\mathcal{P}_\kappa(\kappa)$. For every pair of structures $\mathcal{A}$ and $\mathcal{B}$ with domain κ and $\alpha<\kappa$, the $EF_\omega^\kappa(\mathcal{A}\restriction_\alpha,\mathcal{B}\restriction_\alpha)$ is a game played by the players **I** and **II** as follows.*
*In the n-th move, first **I** choose an ordinal $\beta_n<\alpha$ such that $X_{\beta_n}\subset\alpha$, $X_{\beta_{n-1}}\subseteq X_{\beta_n}$, and then **II** an ordinal $\theta_n<\alpha$ such that $dom(f_{\theta_n}),rang(f_{\theta_n})\subset\alpha$, $X_{\beta_n}\subseteq dom(f_{\theta_n})\cap rang(f_{\theta_n})$ and $f_{\theta_{n-1}}\subseteq f_{\theta_n}$ (if $n=0$ then $X_{\beta_{n-1}}=\emptyset$ and $f_{\theta_{n-1}}=\emptyset$). The game finishes after ω moves. The player **II** wins if $\cup_{i<\omega}f_{\theta_i}:A\restriction_\alpha\to B\restriction_\alpha$ is a partial isomorphism, otherwise the player **I** wins.*

We write $\mathbf{I}\uparrow \mathrm{EF}^\kappa_\omega(\mathcal{A}\restriction_\alpha,\mathcal{B}\restriction_\alpha)$ if $\mathbf{I}$ has a winning strategy in the game $\mathrm{EF}^\kappa_\omega(\mathcal{A}\restriction_\alpha,\mathcal{B}\restriction_\alpha)$. We write $\mathbf{II}\uparrow \mathrm{EF}^\kappa_\omega(\mathcal{A}\restriction_\alpha,\mathcal{B}\restriction_\alpha)$ if $\mathbf{II}$ has a winning strategy.

Lemma 4.8 ([9], Lemma 2.4). *If $\mathcal{A}$ and $\mathcal{B}$ are structures with domain κ, then the following hold:*

- $\mathbf{II}\uparrow EF^\kappa_\omega(\mathcal{A}\restriction\kappa,\mathcal{B}\restriction\kappa) \iff \mathbf{II}\uparrow EF^\kappa_\omega(\mathcal{A}\restriction_\alpha,\mathcal{B}\restriction_\alpha)$ *for club-many α.*
- $\mathbf{I}\uparrow EF^\kappa_\omega(\mathcal{A}\restriction\kappa,\mathcal{B}\restriction\kappa) \iff \mathbf{I}\uparrow EF^\kappa_\omega(\mathcal{A}\restriction_\alpha,\mathcal{B}\restriction_\alpha)$ *for club-many α.*

Proof. It is easy to see that if $\sigma:\kappa^{<\omega}\to\kappa$ is a winning strategy for $\mathbf{II}$ in the game $\mathrm{EF}^\kappa_\omega(\mathcal{A}\restriction\kappa,\mathcal{B}\restriction\kappa)$, then $\sigma\restriction\alpha^{<\alpha}$ is a winning strategy for $\mathbf{II}$ in the game $\mathrm{EF}^\kappa_\omega(\mathcal{A}\restriction_\alpha,\mathcal{B}\restriction_\alpha)$ if $\sigma[\alpha^{<\alpha}]\subseteq\alpha$. So $\mathbf{II}\uparrow \mathrm{EF}^\kappa_\omega(\mathcal{A}\restriction_\alpha,\mathcal{B}\restriction_\alpha)$ for α a closed point of σ.

We conclude that if $\mathbf{II}\uparrow \mathrm{EF}^\kappa_\omega(\mathcal{A}\restriction\kappa,\mathcal{B}\restriction\kappa)$, then $\mathbf{II}\uparrow \mathrm{EF}^\kappa_\omega(\mathcal{A}\restriction_\alpha,\mathcal{B}\restriction_\alpha)$ for club-many α. The same holds for $\mathbf{I}$. To show the other direction, notice that $\mathrm{EF}^\kappa_\omega(\mathcal{A}\restriction\kappa,\mathcal{B}\restriction\kappa)$ is a determined game, so if $\mathbf{II}$ doesn't have a winning strategy, then $\mathbf{I}$ has a winning strategy. Therefore, if $\mathbf{II}$ doesn't have a winning strategy in the game $\mathrm{EF}^\kappa_\omega(\mathcal{A}\restriction\kappa,\mathcal{B}\restriction\kappa)$, then $\mathbf{I}\uparrow \mathrm{EF}^\kappa_\omega(\mathcal{A}\restriction_\alpha,\mathcal{B}\restriction_\alpha)$ for club-many α, and $\mathbf{II}$ cannot have a winning strategy in $\mathrm{EF}^\kappa_\omega(\mathcal{A}\restriction_\alpha,\mathcal{B}\restriction_\alpha)$ for club-many α. □

The reason to introduce these games is that we can characterize classifiable theories with these games.

Theorem 4.9 ([19], XIII Theorem 1.4). *If T is a classifiable theory, then every two models of T that are $L_{\infty,\kappa}$-equivalent are isomorphic.*

Theorem 4.10 ([2], Theorem 10). *$L_{\infty,\kappa}$-equivalence is equivalent to EF^κ_ω-equivalence.*

From these two theorems we know that if T is a classifiable theory, then for any $\mathcal{A}$ and $\mathcal{B}$ models of T with domain κ,

$$\mathbf{II}\uparrow \mathrm{EF}^\kappa_\omega(\mathcal{A},\mathcal{B}) \iff \mathcal{A}\cong\mathcal{B}$$
$$\mathbf{I}\uparrow \mathrm{EF}^\kappa_\omega(\mathcal{A},\mathcal{B}) \iff \mathcal{A}\not\cong\mathcal{B}.$$

From the previous Lemma we know the following two hold for any $\mathcal{A}$ and $\mathcal{B}$ models of a classifiable theory (with domain κ):

- $\mathcal{A}\cong\mathcal{B}\iff \mathbf{II}\uparrow \mathrm{EF}^\kappa_\omega(\mathcal{A}\restriction_\alpha,\mathcal{B}\restriction_\alpha)$ for club-many α.
- $\mathcal{A}\not\cong\mathcal{B}\iff \mathbf{I}\uparrow \mathrm{EF}^\kappa_\omega(\mathcal{A}\restriction_\alpha,\mathcal{B}\restriction_\alpha)$ for club-many α.

Theorem 4.11 ([2], Theorem 70). *If T is a classifiable theory, then $\cong_T$ is $\Delta^1_1(\kappa)$.*

Proof. Notice that the $\mathrm{EF}^\kappa_\omega$ game can be coded as a κ-Borel* game taking at the leaves the open sets given by partial isomorphisms. □

Theorem 4.12 ([2], Theorem 69). *Suppose $\kappa>2^\omega$. If T is a classifiable shallow theory, then $\cong_T$ is κ-Borel.*

Theorem 4.13 ([2], Theorem 71). *If T is unstable, or superstable with OTOP, or superstable with DOP and $\kappa>\omega_1$, then $\cong_T$ is not a $\Delta^1_1(\kappa)$ equivalence relation.*

Definition 4.14. *Let us define the following hierarchy.*

- $\Sigma^0_1=\{X\subseteq 2^\kappa\mid X \textit{ is open}\}$
- $\Pi^0_1=\{X\subseteq 2^\kappa\mid X \textit{ is closed}\}$
- $\Sigma^0_\alpha=\{\bigcup_{\gamma<\kappa}A_\gamma\mid A_\gamma\in\bigcup_{1\leq\beta<\alpha}\Pi^0_\beta\}$
- $\Pi^\alpha_0=\{2^\kappa\backslash X\mid X\in\Sigma^0_\alpha\}$

Notice that κ-Borel$=\bigcup_{\alpha<\kappa^+}\Sigma^0_\alpha$. The smallest ordinal α such that $A\in\Sigma^0_\alpha\cup\Pi^0_\alpha$ is called the Borel rank of A and denoted by $rk_B(A)$. Given a theory T, let us denote by $B(\kappa,T)$ the rank $rk_B(\cong_T)$.

Theorem 4.15 ([13], Theorem 1.9 Descriptive Main Gap). *Let $\kappa>2^\omega$. If T is classifiable shallow of depth α, then $B(\kappa,T)\leq 4\alpha$.*

Notice that under GCH, for all $\gamma,\delta\geq\omega_1$ such that $|\gamma|>|\delta|$, $\kappa=\aleph_{\gamma+\delta}$ satisfies

$$I(T,\aleph_{\gamma+\delta})\leq\beth_{\omega_1}(|\ \gamma+\delta\ |)<\aleph_{\gamma+\delta}$$

Theorem 4.16 ([13], Proposition 6.7). *Let $\kappa=\aleph_\gamma$ be such that $\beth_{\omega_1}(|\ \gamma\ |)\leq\kappa$. Suppose T_1 is a classifiable shallow and T_2 not. Then $\cong_{T_1}\hookrightarrow_c\cong_{T_2}$.*

Lemma 4.17 ([7], Lemma 2)**.** *Let $\mu < \kappa$ is a regular cardinal and $S^\kappa_\mu = \{\alpha < \kappa \mid cf(\alpha) = \mu\}$. Assume T is a classifiable theory and $\mu < \kappa$ is a regular cardinal. If $\diamondsuit_\kappa(S^\kappa_\mu)$ holds then $\cong_T$ is continuously reducible to $=^2_\mu$.*

Proof. Let $\{D_\alpha \mid \alpha \in X\}$ be a sequence testifying $\diamondsuit_\kappa(S^\kappa_\mu)$ and define the function $\mathcal{F}\colon 2^\kappa \to 2^\kappa$ by

$$\mathcal{F}(\eta)(\alpha) = \begin{cases} 1 & \text{if } \alpha \in S^\kappa_\mu \cap C_\pi \cap C_{EF},\ \mathbf{II} \uparrow EF^\kappa_\omega(\mathcal{A}_\eta \restriction_\alpha, \mathcal{A}_{S_\alpha}) \text{ and } \mathcal{A}_\eta \restriction_\alpha \models T \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 4.2. $\eta \cong_T \xi$ *if and only* $\mathcal{F}(\eta) =^2_\mu \mathcal{F}(\xi)$.

□

Theorem 4.18 ([2], Theorem 87)**.** *Suppose that for all $\gamma < \kappa$, $\gamma^\omega < \kappa$ and T is a stable unsuperstable countable theory. Then $=^2_\omega \hookrightarrow_c \cong_T$.* □

Theorem 4.19 ([2], Theorem 79)**.** *Suppose that $\kappa = \lambda^+ = 2^\lambda$ and $\lambda^{<\lambda} = \lambda$.*

1. *If T is unstable or superstable with OTOP, then $=^2_\lambda \hookrightarrow_c \cong_T$.*
2. *If $\lambda \geq 2^\omega$ and T is superstable with DOP, then $=^2_\lambda \hookrightarrow_c \cong_T$.* □

Theorem 4.20 ([7], Theorem 7)**.** *Suppose $\kappa = \lambda^+$, $2^\lambda > 2^\omega$ and $\lambda^{<\lambda} = \lambda$. The following is consistent. If T_1 is classifiable and T_2 is not. Then there is an embedding of $(\mathcal{P}(\kappa), \subseteq)$ to $(B^*(T_1, T_2), \hookrightarrow_B)$, where $B^*(T_1, T_2)$ is the set of all κ-Borel* equivalence relations strictly between $\cong_{T_1}$ and $\cong_{T_2}$.*

From the results of the previous section in L, we obtain the following dichotomy.

Theorem 4.21 ([8], Theorem 4.11)**.** *($V = L$) Suppose that κ is the successor of a regular uncountable cardinal λ. If T is a countable first-order theory in a countable vocabulary, not necessarily complete, then one of the following holds:*

- *$\cong_T$ is Δ^1_1;*
- *$\cong_T$ is Σ^1_1-complete.*

Theorem 4.22 (Friedman-Hyttinen-Kulikov, [2] Theorem 77)**.** *If a first order countable complete theory over a countable vocabulary T is classifiable, then $=^2_\omega \not\hookrightarrow_c \cong_T$.*

Colored Ordered Trees

To study the non-classifiable theories we need to introduce the coloured trees. Coloured trees are very useful to reduce $=^\kappa_\mu$ or $=^2_\mu$ to $\cong_T$, for certain μ and nonclassifiable theory T (see [2], [5], [9], [17]). In [2] and [5] the coloured trees used had height $\omega + 2$ and were used to study the case when κ is a successor cardinal. In [9] the coloured trees had height $\omega + 2$ and were used to study the case when κ is an inaccessible cardinal. In these lectures we will use the coloured trees of [17], i.e. trees of uncountable height and κ inaccessible. Given a tree t, for every $x \in t$ we denote the order type of $\{y \in t | y < x\}$. Let us define $t_\alpha = \{x \in t | ht(x) = \alpha\}$ and $t_{<\alpha} = \cup_{\beta<\alpha} t_\beta$, and denote by $x \restriction \alpha$ the unique $y \in t$ such that $y \in t_\alpha$ and $y \leq x$. If $x, y \in t$ and $\{z \in t | z < x\} = \{z \in t | z < y\}$, then we say that x and y are $\sim$-related, $x \sim y$, and we denote by $[x]$ the equivalence class of x for $\sim$. An α, β-tree is a tree t with the following properties:

- $|[x]| < \alpha$ for every $x \in t$.
- All the branches have order type less than β in t.
- t has a unique root.
- If $x, y \in t$, x and y has no immediate predecessors and $x \sim y$, then $x = y$.

Definition 4.23. *Let λ be an uncountable cardinal. A coloured tree is a pair (t, c), where t is a κ^+, $(\lambda+2)$-tree and c is a map $c : t_\lambda \to \kappa \backslash \{0\}$.*

Definition 4.24. *Let (t, c) be a coloured tree, suppose $(I_\alpha)_{\alpha<\kappa}$ is a collection of subsets of t that satisfies:*

- *for each $\alpha < \kappa$, I_α is a downward closed subset of t.*
- *$\bigcup_{\alpha<\kappa} I_\alpha = t$.*
- *if $\alpha < \beta < \kappa$, then $I_\alpha \subset I_\beta$.*

- *if γ is a limit ordinal, then $I_\gamma = \bigcup_{\alpha<\gamma} I_\alpha$.*
- *for each $\alpha < \kappa$ the cardinality of I_α is less than κ.*

We call $(I_\alpha)_{\alpha<\kappa}$ a filtration of t.

Definition 4.25. *Let t be a coloured tree and $\mathcal{I} = (I_\alpha)_{\alpha<\kappa}$ a filtration of t. Define $H_{\mathcal{I},t} \in \kappa^\kappa$ as follows. Fix $\alpha < \kappa$. Let B_α be the set of all $x \in t_\lambda$ that are not in I_α, but $x \restriction \theta \in I_\alpha$ for all $\theta < \lambda$.*

- *If B_α is non-empty and there is β such that for all $x \in B_\alpha$, $c(x) = \beta$, then let $H_{\mathcal{I},t}(\alpha) = \beta$*
- *Otherwise let $H_{\mathcal{I},t}(\alpha) = 0$*

We will call a filtration good if for every α, $B_\alpha \neq \emptyset$ implies that c is constant on B_α.

Lemma 4.26 ([17])**.** *Suppose (t_0, c_0) and (t_1, c_1) are isomorphic coloured trees, and $\mathcal{I} = (I_\alpha)_{\alpha<\kappa}$ and $\mathcal{J} = (J_\alpha)_{\alpha<\kappa}$ are good filtrations of (t_0, c_0) and (t_1, c_1) respectively. Then $H_{\mathcal{I},t_0} =^\kappa_\lambda H_{\mathcal{J},t_1}$*

Proof. Let $F : (t_0, c_0) \to (t_1, c_1)$ be a coloured tree isomorphism. Define $F\mathcal{I} = (F[I_\alpha])_{\alpha<\kappa}$. It is easy to see that $F[I_\alpha]$ is a downward closed subset of t_1. Clearly $F[I_\alpha] \subset F[I_\beta]$ when $\alpha < \beta$ and for γ a limit ordinal, $\cup_{\alpha<\gamma} F[I_\alpha] = F[I_\gamma]$. If $x \in t_1$ then there exists $y \in t_0$ and $\alpha < \kappa$ such that $F(y) = x$ and $y \in I_\alpha$, therefore $x \in F[I_\alpha]$ and $\cup_{\alpha<\kappa} F[I_\alpha] = t_1$. Since F is an isomorphism, $|F[I_\alpha]| = |I_\alpha| < \kappa$ for every $\alpha < \kappa$. So $F\mathcal{I}$ is a filtration of t_1.
For every α, $B^{\mathcal{I}}_\alpha \neq \emptyset$ implies that $B^{F\mathcal{I}}_\alpha \neq \emptyset$. On the other hand, $\mathcal{I}$ is a good filtration, then when $B^{\mathcal{I}}_\alpha \neq \emptyset$, c_0 is constant on $B^{\mathcal{I}}_\alpha$. Since F is colour preserving, c_1 is constant on $B^{F\mathcal{I}}_\alpha$, we conclude that $F\mathcal{I}$ is a good filtration and $H_{\mathcal{I},t_0}(\alpha) = H_{F\mathcal{I},t_1}(\alpha)$.
Notice that $F[I_\alpha] = J_\alpha$ implies $H_{\mathcal{I},t_0}(\alpha) = H_{\mathcal{J},t_1}(\alpha)$. Therefore it is enough to show that $C = \{\alpha | F[I_\alpha] = J_\alpha\}$ is an λ-club. By the definition of a filtration, for every sequence $(\alpha_i)_{i<\theta}$ in C, cofinal to γ, $J_\gamma = \bigcup_{i<\theta} J_{\alpha_i} = \bigcup_{i<\theta} F[I_{\alpha_i}] = F[I_\gamma]$, so C is closed. To show that C is unbounded, choose $\alpha < \kappa$. Define the succession $(\alpha_i)_{i<\lambda}$ by induction. For $i = 0$, $\alpha_0 = \alpha$. For every limit ordinal γ, when n is odd let $\alpha_{\gamma+n+1}$ be the least ordinal bigger than $\alpha_{\gamma+n}$ such that $F[I_{\alpha_{\gamma+n}}] \subset J_{\alpha_{\gamma+n+1}}$ (such ordinal exists because κ is regular, and $\mathcal{J}$ and $F\mathcal{I}$ are filtrations, specially $|F[I_{\alpha_{\gamma+n}}]| < \kappa$). For every limit ordinal γ, when n is even let $\alpha_{\gamma+n+1}$ be the least ordinal bigger than $\alpha_{\gamma+n}$ such that $J_{\alpha_{\gamma+n}} \subset F[I_{\alpha_{\gamma+n+1}}]$ (such ordinal exists because κ is regular, and $\mathcal{J}$ and $F\mathcal{I}$ are filtrations, specially $|J_{\alpha_n}| < \kappa$). Define $\alpha_\gamma = \bigcup_{i<\gamma} \alpha_i$, then $J_{\alpha_\gamma} = \bigcup_{i<\gamma} J_{\alpha_i} = \bigcup_{i<\gamma} F[I_{\alpha_i}] = F[I_{\alpha_\gamma}]$. Clearly $\bigcup_{i<\lambda} J_{\alpha_i} = \bigcup_{i<\lambda} F[I_{\alpha_i}]$ and $\cup_{i<\lambda} \alpha_i \in C$. □

Order the set $\lambda\times\kappa\times\kappa\times\kappa\times\kappa$ lexicographically, $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) > (\beta_1, \beta_2, \beta_3, \beta_4, \beta_5)$ if for some $1 \leq k \leq 5$, $\alpha_k > \beta_k$ and for every $i < k$, $\alpha_i = \beta_i$. Order the set $(\lambda \times \kappa \times \kappa \times \kappa \times \kappa)^{\leq\lambda}$ as a tree by inclusion.
Define the tree (I_f, d_f) as, I_f the set of all strictly increasing functions from some $\theta \leq \lambda$ to κ and for each η with domain λ, $d_f(\eta) = f(sup(rang(\eta)))$.
For every pair of ordinals α and β, $\alpha < \beta < \kappa$ and $i < \lambda$ define

$$R(\alpha, \beta, i) = \bigcup_{i<j\leq\lambda} \{\eta : [i, j) \to [\alpha, \beta) | \eta \text{ strictly increasing}\}.$$

Definition 4.27. *Assume κ is an inaccessible cardinal. If $\alpha < \beta < \kappa$ and $\alpha, \beta, \gamma \neq 0$, let $\{P^{\alpha,\beta}_\gamma | \gamma < \kappa\}$ be an enumeration of all downward closed subtrees of $R(\alpha, \beta, i)$ for all i, in such a way that each possible coloured tree appears cofinally often in the enumeration. And the tree $P^{0,0}_0$ is (I_f, d_f).*

This enumeration is possible because κ is inaccessible; there are at most $|\bigcup_{i<\lambda} \mathcal{P}(R(\alpha, \beta, i))| \leq \lambda \times \kappa = \kappa$ downward closed coloured subtrees, and at most $\kappa \times \kappa^{<\kappa} = \kappa$ coloured trees.
Denote by $Q(P^{\alpha,\beta}_\gamma)$ the unique ordinal number i such that $P^{\alpha,\beta}_\gamma \subset R(\alpha, \beta, i)$.

Definition 4.28. *Assume κ is an inaccessible cardinal. Define for each $f \in \kappa^\kappa$ the coloured tree (J_f, c_f) by the following construction.*
For every $f \in \kappa^\kappa$ define $J_f = (J_f, c_f)$ as the tree of all $\eta : s \to \lambda \times \kappa^4$, where $s \leq \lambda$, ordered by extension, and such that the following conditions hold for all $i, j < s$:
Denote by η_i, $1 \leq i \leq 5$, the functions from s to κ that satisfies, $\eta(n) = (\eta_1(n), \eta_2(n), \eta_3(n), \eta_4(n), \eta_5(n))$.

1. *$\eta \restriction n \in J_f$ for all $n < s$.*
2. *η is strictly increasing with respect to the lexicographical order on $\lambda \times \kappa^4$.*
3. *$\eta_1(i) \leq \eta_1(i + 1) \leq \eta_1(i) + 1$.*
4. *$\eta_1(i) = 0$ implies $\eta_2(i) = \eta_3(i) = \eta_4(i) = 0$.*

5. $\eta_2(i) \geq \eta_3(i)$ *implies* $\eta_2(i) = 0$.
6. $\eta_1(i) < \eta_1(i+1)$ *implies* $\eta_2(i+1) \geq \eta_3(i) + \eta_4(i)$.
7. *For every limit ordinal* α, $\eta_k(\alpha) = sup_{\beta<\alpha}\{\eta_k(\beta)\}$ *for* $k \in \{1, 2\}$.
8. $\eta_1(i) = \eta_1(j)$ *implies* $\eta_k(i) = \eta_k(j)$ *for* $k \in \{2, 3, 4\}$.
9. *If for some* $k < \lambda$, $[i, j) = \eta_1^{-1}\{k\}$, *then*

$$\eta_5 \restriction [i, j) \in P^{\eta_2(i),\eta_3(i)}_{\eta_4(i)}.$$

Note that 7 implies $Q(P^{\eta_2(i),\eta_3(i)}_{\eta_4(i)}) = i$.

10. *If* $s = \lambda$, *then either*

(a) *there exists an ordinal number* m *such that for every* $k < m$ $\eta_1(k) < \eta_1(m)$, *for every* $k' \geq m$ $\eta_1(k) = \eta_1(m)$, *and the color of* η *is determined by* $P^{\eta_2(m),\eta_3(m)}_{\eta_4(m)}$:

$$c_f(\eta) = c(\eta_5 \restriction [m, \lambda))$$

where c *is the colouring function of* $P^{\eta_2(m),\eta_3(m)}_{\eta_4(m)}$.

or

(b) *there is no such ordinal* m *and then* $c_f(\eta) = f(sup(rang(\eta_5)))$.

Lemma 4.29 ([17])**.** *Assume* κ *is an inaccessible cardinal, then for every* $f, g \in \kappa^\kappa$ *the following holds*

$$f \ =^\kappa_\lambda \ g \Leftrightarrow J_f \cong J_g$$

Proof. By Lemma 2.4, it is enough to prove the following properties of J_f

1. There is a good filtration $\mathcal{I}$ of J_f, such that $H_{\mathcal{I},J_f} \ =^\kappa_\lambda \ f$.
2. If $f \ =^\kappa_\lambda \ g$, then $J_f \cong J_g$.

Notice that for any $k \in rang(\eta_1)$ if $\eta_5 \restriction [i, j) \in P^{\eta_2(i),\eta_3(i)}_{\eta_4(i)}$, when $[i, j) = \eta_1^{-1}(k)$ and if $i + 1 < j$, then $\eta_5 \restriction [i, j)$ is strictly increasing. If $\eta_1(i) < \eta_1(i+1)$, by Definition 2.6 item 6, $\eta_2(i+1) \geq \eta_3(i) + \eta_4(i)$, so $\eta_5(i) < \eta_3(i) \leq \eta_2(i+1) \leq \eta_5(i+1)$. If α is a limit ordinal, by Definition 2.6 items 7 and 8, $\eta_5(\beta) < \eta_2(\beta+1) < \eta_2(\alpha) \leq \eta_5(\alpha)$ it holds for every $\beta < \alpha$. Thus η_5 is strictly increasing. If $\eta \restriction n \in J_f$ for every n, then $\eta \in J_f$. Clearly every maximal branch has order type $\lambda+1$, every chain $\eta \restriction 1 \subset \eta \restriction 2 \subset \eta \restriction 3 \subseteq \cdots$ of any length, has a unique limit in the tree, and every element in t_θ, $\theta < \lambda$, has an infinite number of successors (at most κ), therefore $J_f \in CT^\lambda_*$. For each $\alpha < \kappa$ define J^α_f as

$$J^\alpha_f = \{\eta \in J_f | rang(\eta) \subset \lambda \times (\beta)^4 \text{ for some } \beta < \alpha\}.$$

Suppose $rang(\eta_1) = \lambda$. As it was mentioned before, η_5 is increasing and $sup(rang(\eta_3)) \geq sup(rang(\eta_5)) \geq sup(rang(\eta_2))$. By Definition 2.6 item 6 $sup(rang(\eta_2)) \geq sup(rang(\eta_3))$ and $sup(rang(\eta_2)) \geq sup(rang(\eta_4))$, this lead us to

$$sup(rang(\eta_4)) \leq sup(rang(\eta_3)) = sup(rang(\eta_5)) = sup(rang(\eta_2)). \tag{1}$$

When $\eta \restriction k \in J^\alpha_f$ holds for every $k \in \lambda$, it can be concluded that $sup(rang(\eta_5)) \leq \alpha$, if in addition $\eta \notin J^\alpha_f$, then

$$sup(rang(\eta_5)) = \alpha. \tag{2}$$

Claim 4.30. *Suppose* $\xi \in J^\alpha_f$ *and* $\eta \in J_f$. *If* $dom(\xi)$ *a successor ordinal smaller than* λ, $\xi \subsetneq \eta$ *and for every* k *in* $dom(\eta)\backslash dom(\xi)$, $\eta_1(k) = \xi_1(max(dom(\xi)))$ *and* $\eta_1(k) > 0$, *then* $\eta \in J^\alpha_f$.

Proof. Assume $\xi, \eta \in J_f$ are as in the assumption. Let $\beta_i = \xi_i(max(dom(\xi)))$, for $i \in \{2, 3, 4\}$. Since $\xi \in J^\alpha_f$, then there exists $\beta < \alpha$ such that $\beta_2, \beta_3, \beta_4 < \beta$. By Definition 2.6 item 8 for every $k \in dom(\eta)\backslash dom(\xi)$, $\eta_i(k) = \beta_i$ for $i \in \{2, 3, 4\}$. Therefore, by Definition 2.6 item 9 and the definition of $P^{\beta_2,\beta_3}_{\beta_4}$, we conclude $\eta_5(k) < \beta_3 < \beta$, so $\eta \in J^\alpha_f$. □

Claim 4.31. $|J_f| = \kappa$, $\mathcal{J} = (J^\alpha_f)_{\alpha<\kappa}$ *is a good filtration of* J_f *and* $H_{\mathcal{J},J_f} \ =^\kappa_\lambda \ f$

Proof. Clearly $J_f = \cup_{\alpha<\kappa} J_f^\alpha$, J_f^α is a downward closed subset of J_f, and $J_f^\alpha \subset J_f^\beta$ when $\alpha < \beta$. Since κ is inaccessible, we conclude $|J_f^\alpha| < \kappa$ and $|J_f| = \kappa$. Finally, when γ is a limit ordinal

$$\begin{aligned} J_f^\gamma &= \{\eta \in J_f | \exists \beta < \gamma (rang(\eta) \subset \omega \times (\beta)^4)\} \\ &= \{\eta \in J_f | \exists \alpha < \gamma, \exists \beta < \alpha (rang(\eta) \subset \omega \times (\beta)^4)\} \\ &= \bigcup_{\alpha<\gamma} J_f^\alpha \end{aligned}$$

Suppose α has cofinality λ, and $\eta \in J_f \backslash J_f^\alpha$ satisfies $\eta \restriction k \in J_f^\alpha$ for every $k < \lambda$. By the previous claim, η satisfies Definition 2.6 item 10 (a) only if $\eta_1(n) = 0$ for every $n \in \lambda$. So η_1, η_2, η_3 and η_4 are constant zero, and $c_f(\eta) = d_f(\eta_5)$, where d_f is the colouring function of $P_0^{0,0} = I_f$, $c_f(\eta) = f(sup(rang(\eta_5)))$. When η satisfies Definition 2.6 item 10 (b), $c_f(\eta) = f(sup(rang(\eta_5)))$.
In both cases, $c_f(\eta) = f(\alpha)$. Therefore, if $B_\alpha \neq \emptyset$ then c_f is constant on B_α and $\mathcal{J}$ is a good filtration.
By Definition 2.3 and since $\mathcal{J}$ is a good filtration, $H_{\mathcal{J},J_f}(\alpha) = f(\alpha)$. □

Claim 4.32. *If $f =_\lambda^\kappa g$, then $J_f \cong J_g$.*

Proof. Let $C' \subseteq \{\alpha < \kappa | f(\alpha) = g(\alpha)\}$ be an λ-club testifying $f =_\lambda^\kappa g$, and let $C \supset C'$ be the closure of C' under limits. By induction we are going to construct an isomorphism between J_f and J_g.
We define continuous increasing sequences $(\alpha_i)_{i<\kappa}$ of ordinals and $(F_{\alpha_i})_{i<\kappa}$ of partial color-preserving isomorphism from J_f to J_g such that:

a) If i is a successor, then α_i is a successor ordinal and there exists $\beta \in C$ such that $\alpha_{i-1} < \beta < \alpha_i$ and thus if i is a limit, $\alpha_i \in C$.

b) Suppose that $i = \gamma + n$, where γ is a limit ordinal or 0, and $n < \omega$ is even. Then $dom(F_{\alpha_i}) = J_f^{\alpha_i}$.

c) Suppose that $i = \gamma + n$, where γ is a limit ordinal or 0, and $n < \omega$ is odd. Then $rang(F_{\alpha_i}) = J_g^{\alpha_i}$.

d) If $dom(\xi) < \lambda$, $\xi \in dom(F_{\alpha_i})$, $\eta \restriction dom(\xi) = \xi$ and for every $k \geq dom(\xi)$

$$\eta_1(k) = \xi_1(sup(dom(\xi))) \text{ and } \eta_1(k) > 0$$

then $\eta \in dom(F_{\alpha_i})$. Similar for $rang(F_{\alpha_i})$.

e) If $\xi \in dom(F_{\alpha_i})$ and $k < dom(\xi)$, then $\xi \restriction k \in dom(F_{\alpha_i})$.

f) For all $\eta \in dom(F_{\alpha_i})$, $dom(\eta) = dom(F_{\alpha_i}(\eta))$.

For every ordinal α denote by $M(\alpha)$ the ordinal that is order isomorphic to the lexicographic order of $\lambda \times \alpha^4$.

First step (i=0).
Let $\alpha_0 = \beta + 1$ for some $\beta \in C$. Let γ be an ordinal such that there is a coloured tree isomorphism $h : P_\gamma^{0,M(\beta)} \to J_f^{\alpha_0}$ and $Q(P_\gamma^{0,M(\beta)}) = 0$. It is easy to see that such γ exists, by the way our enumeration was chosen.
Since $P_\gamma^{0,M(\beta)}$ and $J_f^{\alpha_0}$ are closed under initial segments, then $|dom(h^{-1}(\eta))| = |dom(\eta)|$. Also both domains are intervals containing zero, therefore $dom(h^{-1}(\eta)) = dom(\eta)$.
Define $F_{\alpha_0}(\eta)$ for $\eta \in J_f^{\alpha_0}$ as follows, let $F_{\alpha_0}(\eta)$ be the function ξ with $dom(\xi) = dom(\eta)$, and for all $\kappa < dom(\xi)$

- $\xi_1(k) = 1$
- $\xi_2(k) = 0$
- $\xi_3(k) = M(\beta)$
- $\xi_4(k) = \gamma$
- $\xi_5(k) = h^{-1}(\eta)(k)$

To check that $\xi \in J_g$, we will check every item of Definition 2.6. Since $rang(F_{\alpha_0}) = \{1\} \times \{0\} \times \{M(\beta)\} \times \{\gamma\} \times P_\gamma^{0,M(\beta)}$, ξ satisfies 1. Also $\xi_5 = h^{-1}(\eta) \in P_\gamma^{0,M(\beta)}$, by definition of $P_\gamma^{\alpha,\beta}$, we now that ξ_5 is strictly increasing with respect to the lexicographic order, then ξ satisfies item 2. Notice that ξ is constant in every component except for ξ_5, therefore ξ satisfies the items 3, 6, 7, 8, 10 (a). Clearly $\xi_1(i) \neq 0$, so ξ satisfies item 4. Since $\xi_2(k) = 0$ for every k, then ξ satisfies 5. Notice that $[0, \lambda) = \xi_1^{-1}(1)$ but $P_{\xi_4(k)}^{\xi_2(k),\xi_3(k)} = P_\gamma^{0,M(\beta)}$ for every k, therefore $\xi_5 \in P_{\xi_4(0)}^{\xi_2(0),\xi_3(0)}$ and ξ satisfies 7.
Let us show that the conditions a)-f) are satisfied, the conditions a) and c) are clearly satisfied. By the way F_{α_0} was defined, $dom(F_{\alpha_0}) = J_f^{\alpha_0}$ and $dom(\eta) = dom(F_{\alpha_0}(\eta))$, these are the conditions b), e) and f). Since

$dom(F_{\alpha_0}) = J_f^{\alpha_0}$, the Claim 2.7.1 implies d) for $dom(F_{\alpha_0})$. For d) with $rang(F_{\alpha_0})$, suppose $\xi \in rang(F_{\alpha_0})$ and $\eta \in J_g$ are as in the assumption. Then $\eta_1(k) = \xi_1(k) = 1$ for every $k < dom(\eta)$, by 8 in J_g we have that $\eta_2(k) = \xi_2(k) = 0$, $\eta_3(k) = \xi_3(k) = M(\beta)$ and $\eta_4(k) = \xi_4(k) = \gamma$ for every $k < dom(\eta)$. By 9 in J_g, $\eta_5 \in P_\gamma^{0,M(\beta)}$ and since $rang(F_{\alpha_0}) = \{1\} \times \{0\} \times \{M(\beta)\} \times \{\gamma\} \times P_\gamma^{0,M(\beta)}$, we can conclude that $\eta \in rang(F_{\alpha_0})$.

Odd successor step.
Suppose that $j < k$ is a successor ordinal such that $j = \beta_j + n_j$ for some limit ordinal (or 0) β_j and an odd integer n_j. Assume α_l and F_{α_l} are defined for every $l < j$ satisfying the conditions a)-f).
Let $\alpha_j = \beta + 1$ where $\beta \in C$ is such that $\beta > \alpha_{j-1}$ and $rang(F_{\alpha_{j-1}}) \subset J_g^\beta$, such a β exists because $|rang(F_{\alpha_{j-1}})| \leq 2^{|\alpha_{j-1}|}$ and κ is strongly inaccessible.
When $\eta \in rang(F_{\alpha_{j-1}})$ has domain $m < \lambda$, define

$$W(\eta) = \{\zeta | dom(\zeta) = [m, s), m < s \leq \lambda, \eta^\frown\langle m, \zeta(m)\rangle \notin rang(F_{\alpha_{j-1}}) \text{ and } \eta^\frown\zeta \in J_g^{\alpha_j}\}$$

with the color function $c_{W(\eta)}(\zeta) = c_g(\eta^\frown\zeta)$ for every $\zeta \in W(\eta)$ with $s = \lambda$. Denote $\xi' = F_{\alpha_{j-1}}^{-1}(\eta)$, $\alpha = \xi_3'(m-1) + \xi_4'(m-1)$ (if m is a limit ordinal, then $\alpha = sup_{\theta<m}\xi_2(\theta)$) and $\theta = \alpha + M(\alpha_j)$. Now choose an ordinal γ_η such that $Q(P_{\gamma_\eta}^{\alpha,\theta}) = m$ and there is an isomorphism $h_\eta : P_{\gamma_\eta}^{\alpha,\theta} \to W(\eta)$. We will define F_{α_j} by defining its inverse such that $rang(F_{\alpha_j}) = J_g^{\alpha_j}$.
Each $\eta \in J_g^{\alpha_j}$ satisfies one of the followings:

(*) $\eta \in rang(F_{\alpha_{j-1}})$.

(**) $\exists m < dom(\eta)(\eta \restriction m \in rang(F_{\alpha_{j-1}}) \wedge \eta \restriction (m+1) \notin rang(F_{\alpha_{j-1}}))$.

(***) $\forall m < dom(\eta)(\eta \restriction (m+1) \in rang(F_{\alpha_{j-1}}) \wedge \eta \notin rang(F_{\alpha_{j-1}}))$.

We define $\xi = F_{\alpha_j}^{-1}(\eta)$ as follows. There are three cases:

Case η satisfies $(*)$.
Define $\xi(n) = F_{\alpha_{j-1}}^{-1}(\eta)(n)$ for all $n < dom(\eta)$.

Case η satisfies $(**)$.
This case is divided in two subcases, when m is limit ordinal and when m is successor ordinal. Let m witnesses (**) for η and suppose m is a successor ordinal. For every $n < dom(\xi)$

- If $n < m$, then $\xi(n) = F_{\alpha_{j-1}}^{-1}(\eta \restriction m)(n)$.
- For every $n \geq m$. Let
  - $\xi_1(n) = \xi_1(m-1) + 1$
  - $\xi_2(n) = \xi_3(m-1) + \xi_4(m-1)$
  - $\xi_3(n) = \xi_2(m) + M(\alpha_j)$
  - $\xi_4(n) = \gamma_{\eta\restriction m}$
  - $\xi_5(n) = h_{\eta\restriction m}^{-1}(\eta \restriction [m, dom(\eta)))(n)$

Note that, $\eta \restriction [m, dom(\eta))$ is an element of $W(\eta \restriction m)$, this makes possible the definition of ξ_5.
Let us check the items of Definition 2.6 to see that $\xi \in J_f$. Clearly item 1 is satisfied. By induction hypothesis, $\xi \restriction m$ is increasing, $\xi_1(m) = \xi_1(m-1) + 1$ so $\xi(m-1) < \xi(m)$, and ξ_k is constant on $[m, \lambda)$ for $k \in \{1, 2, 3, 4\}$, since $h_{\eta\restriction m}^{-1}(\eta) \in P_{\gamma_\eta}^{\alpha,\theta}$, then ξ_5 is increasing, and we conclude that ξ is increasing with respect to the lexicographic order, so ξ satisfies item 2. Also we conclude $\xi_1(i) \leq \xi_1(i+1) \leq \xi_1(i) + 1$, so ξ satisfies item 3. For every $i < \lambda$, $\xi_1(i) = 0$ implies $i < m$, so $\xi(i) = F_{\alpha_{j-1}}^{-1}(\eta \restriction m)(i)$ and by the induction hypothesis ξ satisfies item 4. By the induction hypothesis, $\xi \restriction m \in J_f$, since $\xi_2(n) = \xi_3(m-1) + \xi_4(m-1)$ holds for every $n \geq m$, we conclude that ξ satisfies 5. By the induction hypothesis, for every $i + 1 < m$, $\xi_1(i) < \xi_1(i+1)$ implies $\xi_2(i+1) \geq \xi_3(i) + \xi_4(i)$, on the other hand $\xi_1(i) = \xi_1(j)$ implies $\xi_k(i) = \xi_k(j)$ for $k \in \{2, 3, 4\}$, clearly $\xi_2(m) \geq \xi_3(m-1) + \xi_4(m-1)$ and $\xi_k(i) = \xi_k(i+1)$ for $i \geq m$ and $k \in \{2, 3, 4\}$, then ξ satisfies items 6 and 8.
By the induction hypothesis, $\xi \restriction m \in J_f$, since $\xi_1(n) = \xi_1(m-1) + 1$ and $\xi_2(n) = \xi_3(m-1) + \xi_4(m-1)$ hold for every $n \geq m$, we conclude that ξ satisfies 7. Suppose $[i, j) = \xi_1^{-1}(k)$ for some k in $rang(\xi)$. Either $j < m$ or $m = i$. If $j < m$, by the induction hypothesis $\xi_5 \restriction [i, j) \in P_{\xi_4(i)}^{\xi_2(i),\xi_3(i)}$, if $[i, j) = [m, dom(\xi))$, then $\xi_5 \restriction [i, j) = h_{\eta\restriction m}^{-1}(\eta \restriction [m, dom(\xi))) \in P_{\xi_4(m)}^{\xi_2(m),\xi_3(m)}$, ξ thus satisfies item 9. Since ξ is constant on $[m, \lambda)$, ξ satisfies 10 (a). Finally by item 10 (a) when $dom(\zeta) = \lambda$, $c_f(\xi) = c(\xi_5 \restriction [m, \lambda))$, where c is the color of $P_{\xi_4(m)}^{\xi_2(m),\xi_3(m)}$. Since $\xi_5 \restriction [m, \lambda) = h_{\eta\restriction m}^{-1}(\eta \restriction [m, \lambda))$, $c_f(\xi) = c(h_{\eta\restriction m}^{-1}(\eta \restriction [m, \lambda)))$ and since h is an isomorphism,

$c_f(\xi) = c_{W(\eta\restriction_m)}(\eta \restriction [m, \lambda)) = c_g(\eta)$.

Let m witnesses (**) for η and suppose m is a limit ordinal. For every $n < dom(\xi)$

- If $n < m$, then $\xi(n) = F^{-1}_{\alpha_{j-1}}(\eta \restriction m)(n)$.
- For every $n \geq m$. Let
  - $\xi_1(n) = sup_{\theta<m}\xi_1(\theta)$
  - $\xi_2(n) = sup_{\theta<m}\xi_2(\theta)$
  - $\xi_3(n) = \xi_2(m) + M(\alpha_j)$
  - $\xi_4(n) = \gamma_{\eta\restriction m}$
  - $\xi_5(n) = h^{-1}_{\eta\restriction m}(\eta \restriction [m, dom(\eta)))(n)$

Note that, $\eta \restriction [m, dom(\eta))$ is an element of $W(\eta \restriction m)$, this makes possible the definition of ξ_5.
Let us check the items of Definition 2.6 to see that $\xi \in J_f$. Clearly item 1 is satisfied. By induction hypothesis, $\xi \restriction m$ is increasing, $\xi_1(m) = sup_{\theta<m}\xi_1(\theta)$ so $\xi(\theta) < \xi(m)$ for every $\theta < m$, and ξ_k is constant on $[m, \lambda)$ for $k \in \{1, 2, 3, 4\}$, since $h^{-1}_{\eta\restriction m}(\eta) \in P^{\alpha,\theta}_{\gamma_\eta}$, then ξ_5 is increasing, and we conclude that ξ is increasing with respect to the lexicographic order, so ξ satisfies item 2. Also we conclude $\xi_1(i) \leq \xi_1(i + 1) \leq \xi_1(i) + 1$, so ξ satisfies item 3. For every $i < \lambda$, $\xi_1(i) = 0$ implies $i < m$, so $\xi(i) = F^{-1}_{\alpha_{j-1}}(\eta \restriction m)(i)$ and by the induction hypothesis ξ satisfies item 4. By the induction hypothesis, $\xi \restriction m \in J_f$, since $\xi_2(n) = sup_{\theta<m}\xi_2(\theta)$ holds for every $n \geq m$, we conclude that ξ satisfies 5. By the induction hypothesis, for every $i + 1 < m$, $\xi_1(i) < \xi_1(i + 1)$ implies $\xi_2(i + 1) \geq \xi_3(i) + \xi_4(i)$, on the other hand $\xi_1(i) = \xi_1(j)$ implies $\xi_k(i) = \xi_k(j)$ for $k \in \{2, 3, 4\}$, clearly $\xi_2(m) \geq sup_{\theta<m}\xi_3(\theta)$ and $\xi_k(i) = \xi_k(j)$ for $j, i \geq m$ and $k \in \{2, 3, 4\}$, then ξ satisfies items 6 and 8.
By the induction hypothesis, $\xi \restriction m \in J_f$, since $\xi_1(n) = sup_{\theta<m}\xi_1(\theta)$ and $\xi_2(n) = sup_{\theta<m}\xi_2(\theta)$ hold for every $n \geq m$, we conclude that ξ satisfies 7. Suppose $[i, j) = \xi_1^{-1}(k)$ for some k in $rang(\xi)$. Either $j < m$ or $m = i$, notice that if $i < m < j$, then $\eta \restriction (m + 1) \in rang(F_{\alpha_{j-1}})$). If $j < m$, by the induction hypothesis $\xi_5 \restriction [i, j) \in P^{\xi_2(i),\xi_3(i)}_{\xi_4(i)}$, if $[i, j) = [m, dom(\xi))$, then $\xi_5 \restriction [i, j) = h^{-1}_{\eta\restriction m}(\eta \restriction [m, dom(\xi))) \in P^{\xi_2(m),\xi_3(m)}_{\xi_4(m)}$, ξ thus satisfies item 9. Since ξ is constant on $[m, \lambda)$, ξ satisfies 10 (a). Finally by item 10 (a) when $dom(\zeta) = \lambda$, $c_f(\xi) = c(\xi_5 \restriction [m, \lambda))$, where c is the color of $P^{\xi_2(m),\xi_3(m)}_{\xi_4(m)}$. Since $\xi_5 \restriction [m, \lambda) = h^{-1}_{\eta\restriction m}(\eta \restriction [m, \lambda))$, $c_f(\xi) = c(h^{-1}_{\eta\restriction m}(\eta \restriction [m, \lambda)))$ and since h is an isomorphism, $c_f(\xi) = c_{W(\eta\restriction_m)}(\eta \restriction [m, \lambda)) = c_g(\eta)$.

Case η satisfies $(***)$.
Clearly $dom(\eta) = \lambda$, by the induction hypothesis and condition d), $rang(\eta) = \lambda$, otherwise $\eta \in rang(F_{\alpha_{j-1}})$. Let $F^{-1}_{\alpha_j}(\eta) = \xi = \cup_{n<\lambda}F^{-1}_{\alpha_{j-1}}(\eta \restriction n)$, by the induction hypothesis, ξ is well defined. Since for every $n < \lambda$, $\xi \restriction n \in J_f$, then $\xi \in J_f$. Let us check that $c_f(\xi) = c_g(\eta)$. First note that $\xi \notin J^{\alpha_{j-1}}_f$, otherwise by the induction hypothesis f),

$$F_{\alpha_{j-1}}(\xi) = \bigcup_{n<\lambda} F_{\alpha_{j-1}}(\xi \restriction n) = \bigcup_{n<\lambda} \eta \restriction n = \eta$$

giving us $\eta \in rang(F_{\alpha_{j-1}})$. By the equation (2), $sup(rang(\xi_5)) = \alpha_{j-1}$ and ξ satisfies item 10 b) in J_f, therefore $c_f(\xi) = f(\alpha_{j-1})$. Also by the definition of J^α_f and since $\xi \restriction n \in J^{\alpha_{j-1}}_f$ for every $n < \lambda$, α_{j-1} is a limit ordinal and by condition a), $j - 1$ is a limit ordinal and $\alpha_{j-1} \in C$. The conditions b) and c) ensure $rang(F_{\alpha_{j-1}}) = J^{\alpha_{j-1}}_f$. This implies, $\eta \notin J^{\alpha_{j-1}}_f$. By the equation (2), $sup(rang(\eta_5)) = \alpha_{j-1}$. Therefore α_{j-1} has cofinality λ, $\alpha_{j-1} \in C'$ and $f(\alpha_{j-1}) = g(\alpha_{j-1})$. By item 10 b) in J_g, $c_g(\eta) = g(\alpha_{j-1}) = f(\alpha_{j-1}) = c_f(\xi)$.

Next we show that F_{α_i} is a color preserving partial isomorphism. We already showed that F_{α_i} preserve the colors, so we only need to show that

$$\eta \subsetneq \xi \Leftrightarrow F^{-1}_{\alpha_i}(\eta) \subsetneq F^{-1}_{\alpha_i}(\xi). \tag{3}$$

From left to right.

When $\eta, \xi \in rang(F_{\alpha_{i-1}})$, the induction hypothesis implies (3) from left to right. If $\eta \in rang(F_{\alpha_{i-1}})$ and $\xi \notin rang(F_{\alpha_{i-1}})$, the construction implies (3) from left to right. Let us assume $\eta, \xi \notin rang(F_{\alpha_{i-1}})$, then η, ξ satisfy (**). Let m_1 and m_2 be the respective ordinal numbers that witness (**) for η and ξ, respectively. Notice that $m_2 < dom(\eta)$, otherwise, $\eta \in rang(F_{\alpha_{i-1}})$. If $m_1 < m_2$, clearly $\eta \in rang(F_{\alpha_{i-1}})$ what is not the case. A similar argument shows that $m_2 < m_1$ cannot hold. We conclude that $m_1 = m_2$ and by the construction of F_{α_i}, $F^{-1}_{\alpha_i}(\eta) \subsetneq F^{-1}_{\alpha_i}(\xi)$.

From right to left.

When $\eta, \xi \in rang(F_{\alpha_{i-1}})$, the induction hypothesis implies (3) from right to left. If $\eta \in rang(F_{\alpha_{i-1}})$ and $\xi \notin rang(F_{\alpha_{i-1}})$, the construction implies (3) from right to left. Let us assume $\eta, \xi \notin rang(F_{\alpha_{i-1}})$, then η, ξ satisfy (**). Let m_1 and m_2 be the respective ordinal numbers that witness (**) for η and ξ, respectively.

Notice that $m_2 < dom(\eta)$, otherwise, $F^{-1}_{\alpha_i}(\eta) = F^{-1}_{\alpha_{i-1}}(\eta)$ and $\eta \in rang(F_{\alpha_{i-1}})$. Let us denote by θ the inverse map $F^{-1}_{\alpha_i}$ (e.g. $\theta(\zeta) = F^{-1}_{\alpha_i}(\zeta)$), and the first component by θ_1 (e.g. $\theta_1(\zeta) = F^{-1}_{\alpha_i}(\zeta)_1$).
If $m_1 < m_2$ and m_2 is a successor ordinal, then

$$\begin{array}{rcl} \theta_1(\eta)(m_2-1) & = & (\theta(\xi)\restriction_{m_2})_1(m_2-1) \\ & < & \theta_1(\xi\restriction_{m_2})(m_2-1)+1 \\ & = & \theta_1(\eta)(m_2) \\ & = & \theta_1(\eta)(m_2-1). \end{array}$$

If $m_1 < m_2$ and m_2 is a limit ordinal, then

$$\begin{array}{rcl} \forall\gamma \in [m_1, m_2) \;\; \theta_1(\eta)(\gamma) & = & (\theta(\xi)\restriction_{m_2})_1(\gamma) \\ & < & sup_{n<m_2}\theta_1(\xi\restriction_{m_2})(n) \\ & = & \theta_1(\eta)(m_2) \\ & = & \theta_1(\eta)(\gamma). \end{array}$$

This cannot hold. A similar argument shows that $m_2 < m_1$ cannot hold. We conclude that $m_1 = m_2$.
By the induction hypothesis $F^{-1}_{\alpha_{i-1}}(\eta \restriction m_1) = F^{-1}_{\alpha_{i-1}}(\xi \restriction m_2)$ implies $\eta \restriction m_1 = \xi \restriction m_2$ (also implies $h_{\eta\restriction m_1} = h_{\xi\restriction m_2}$). Since $F^{-1}_{\alpha_{i-1}}(\eta \restriction m_1)(n) = F^{-1}_{\alpha_i}(\eta)(n)$ for all $n < m_1$, we only need to prove that $\eta \restriction [m_1, dom(\eta)) \subsetneq \xi \restriction [m_2, dom(\xi))$. But $h_{\eta\restriction m_1}$ is an isomorphism and $F^{-1}_{\alpha_i}(\eta)_5(n) = F^{-1}_{\alpha_i}(\xi)_5(n)$ for every $n \geq m_1$, so $h^{-1}_{\eta\restriction m_1}(\eta \restriction [m_1, dom(\eta)))(n) = h^{-1}_{\xi\restriction m_2}(\xi \restriction [m_2, dom(\xi)))(n)$. Therefore $\eta \restriction [m_1, dom(\eta)) \subsetneq \xi \restriction [m_2, dom(\xi))$.

Let us check that this three constructions satisfy the conditions a)-f).
When i is a successor we have $\alpha_{i-1} < \beta < \alpha_i = \beta+1$ for some $\beta \in C$, this is the condition a). Clearly the three cases satisfy b). We defined $F^{-1}_{\alpha_i}$ according to (*), (**), or (***); since every $\eta \in J^{\alpha_j}_g$ satisfies one of these, we conclude $rang(F_{\alpha_i}) = J^{\alpha_j}_g$ which is the condition c).
Let us show that the F_{α_i} satisfy condition d). Let ξ and η be as in the assumptions of condition d) for domain. Notice that if $\xi \in dom(F_{\alpha_{i-1}})$ then the induction hypothesis ensure that $\eta \in dom(F_{\alpha_i})$. Suppose $\xi \notin dom(F_{\alpha_{i-1}})$, then $F_{\alpha_i}(\xi) \notin rang(F_{\alpha_{i-1}})$. Since $dom(\xi) < \lambda$, so $F_{\alpha_i}(\xi)$ satisfies (**). Let m be the number witnessing it. If m is a limit ordinal, then $dom(\xi) \geq m+1$, therefore $\xi \restriction m+1 \in J^{\alpha_i}_f$ and by Claim 2.7.1 $\eta \in J^{\alpha_i}_f$. If m is a successor ordinal, then $\xi \in J^{\alpha_i}_f$ and by Claim 2.7.1 $\eta \in J^{\alpha_i}_f$. By item 8 in $J^{\alpha_i}_f$, η_k is constant on $[m, dom(\eta))$ for $k \in \{2,3,4\}$, now by Definition 2.6 item 9 in $J^{\alpha_i}_f$, $\eta_5 \restriction [m, dom(\eta)) \in P^{\alpha,\beta}_{\gamma_{\xi\restriction m}}$. Let $\zeta = h_{\xi\restriction m}(\eta_{[m,dom(\eta))})$, then $\eta = F^{-1}_{\alpha_i}(F_{\alpha_i}(\xi \restriction m)^\frown\zeta)$ and $\eta \in dom(F_{\alpha_i})$.
Using the same argument, the condition d) can be proved.
For the conditions e) and f), notice that ξ was constructed such that $dom(\xi) = dom(\eta)$ and $\xi \restriction k \in dom(F_{\alpha_i})$ which are these conditions.

Even successor step.
Suppose that $j < k$ is a successor ordinal such that $j = \beta_j + n_j$ for some limit ordinal (or 0) β_j and an even integer n_j. Assume α_l and F_{α_l} are defined for every $l < j$ satisfying conditions a)-f).
Let $\alpha_j = \beta+1$ where $\beta \in C$ such that $\beta > \alpha_{j-1}$ and $dom(F_{\alpha_{j-1}}) \subset J^\beta_f$, such a β exists because $|dom(F_{\alpha_{j-1}})| \leq 2^{|\alpha_{j-1}|}$ and κ is strongly inaccessible. The construction of F_{α_j} such that $dom(F_{\alpha_j}) = J^{\alpha_i}_f$ follows as in the odd successor step, with the equivalent definitions for $dom(F_{\alpha_j})$ and $J^{\alpha_i}_f$. Notice that for every $\eta \in J^{\alpha_j}_f$, there are only the following cases:

(*) $\eta \in dom(F_{\alpha_{j-1}})$.

(**) $\exists m < dom(\eta)(\eta \restriction m \in dom(F_{\alpha_{j-1}}) \wedge \eta \restriction (m+1) \notin dom(F_{\alpha_{j-1}}))$.

Limit step.
Assume j is a limit ordinal. Let $\alpha_j = \cup_{i<j}\alpha_i$ and $F_{\alpha_j} = \cup_{i<j}F_{\alpha_i}$, clearly $F_{\alpha_j}: J^{\alpha_j}_f \to J_g$ and satisfies condition c). Since for i successor, α_i is the successor of an ordinal in C, then $\alpha_j \in C$ and satisfies the condition a). Also F_{α_j} is a partial isomorphism. Remember that $\cup_{i<j}J^{\alpha_i}_f = J^{\alpha_j}_f$, the same for J_g. By the induction hypothesis and the conditions b) and c) for $i < j$, we have $dom(F_{\alpha_j}) = J^{\alpha_j}_f$ (this is the condition b)) and $rang(F_{\alpha_j}) = J^{\alpha_j}_g$. This and Claim 2.7.1 ensure that condition d) is satisfied. By the induction hypothesis, for every $i < j$, F_{α_i} satisfies conditions e) and f), then F_{α_j} satisfies conditions e) and f). □

Define $F = \cup_{i<\kappa}F_{\alpha_i}$, clearly, it is an isomorphism between J_f and J_g. □

Definition 4.33. *Let K^γ_{tr} be the class of models $(A, \prec, (P_n)_{n\leq\gamma}, <, \wedge)$, where:*

1. *there is a linear order $(I, <_I)$ such that $A \subseteq I^{\leq\gamma}$;*

2. *A is closed under initial segment;*

3. *$\prec$ is the initial segment relation;*

4. *$\wedge(\eta, \xi)$ is the maximal common initial segment of η and ξ;*

5. *let $lg(\eta)$ be the length of η (i.e. the domain of η) and $P_n = \{\eta \in A \mid lg(\eta) = n\}$ for $n \leq \gamma$;*

6. *for every $\eta \in A$ with $lg(\eta) < \gamma$, define $Suc_A(\eta)$ as $\{\xi \in A \mid \eta \prec \xi \ \&\ lg(\xi) = lg(\eta) + 1\}$. If $\xi < \zeta$, then there is $\eta \in A$ such that $\xi, \zeta \in Suc_A(\eta)$;*

7. *$\eta < \xi$ if and only if either $\eta \prec \xi$ or there is $\zeta \in A$ and x, y such that $\eta = \zeta^\frown\langle x\rangle$, $\xi = \zeta^\frown\langle y\rangle$, and $x <_I y$;*

8. *If η and ξ have no immediate predecessor and $\{\zeta \in A \mid \zeta \prec \eta\} = \{\zeta \in A \mid \zeta \prec \xi\}$, then $\eta = \xi$.*

An ordered tree is an element of K^γ_{tr}. An ordered coloured tree is a tree $T \in K^\gamma_{tr}$ with a color function $c : t_\gamma \to \beta$. For any $\mathcal{L}$-structure M we denote by at the set of atomic formulas of $\mathcal{L}$ and by bs the set of basic formulas of $\mathcal{L}$ (atomic formulas and negation of atomic formulas). For all $\mathcal{L}$-structure M, $a \in M$, and $B \subseteq M$ we define

$$tp_{bs}(a, B, M) = \{\varphi(x, b) \mid M \models \varphi(a, b), \varphi \in bs, b \in B\}.$$

In the same way $tp_{at}(a, B, M)$ is defined.

Definition 4.34. *Let I be a linear order of size κ. We say that I is κ-colorable if there is a function $F : I \to \kappa$ such that for all $B \subseteq I$, $|B| < \kappa$, $b \in I \backslash B$, and $p = tp_{bs}(b, B, I)$ such that the following hold: For all $\alpha \in \kappa$, $|\{a \in I \mid a \models p \ \&\ F(a) = \alpha\}| = \kappa$.*

Theorem 4.35 ([15], Theorem 2.25)**.** *There is a $(< \kappa, bs)$-stable (κ, bs, bs)-nice κ-colorable linear order.*

Notice that $J^0_f = \{\emptyset\}$ and $dom(\emptyset) = 0$. Let us denote by $Acc(\kappa) = \{\alpha < \kappa \mid \alpha = 0 \text{ or } \alpha \text{ is a limit ordinal}\}$. For all $\alpha \in Acc(\kappa)$ and $\eta \in J^\alpha_f$ with $dom(\eta) = m < \omega$ define

$$W^\alpha_\eta = \{\zeta \mid dom(\zeta) = [m, s), m \leq s \leq \omega, \eta^\frown\zeta \in J^{\alpha+\omega}_f, \eta^\frown(\zeta \restriction \{m\}) \notin J^\alpha_f\}.$$

Notice that by the way J_f was constructed, for every $\eta \in J_f$ with finite domain and $\alpha < \kappa$, the set

$$\{(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5) \in (\omega \times \kappa^4) \backslash (\omega \times \alpha^4) \mid \eta^\frown(\theta_1, \theta_2, \theta_3, \theta_4, \theta_5) \in J^{\alpha+\omega}_f\}$$

is either empty or has size ω. Let σ^α_η be an enumeration of this set, when this set is not empty.

Let us denote by $\mathcal{T} = (\kappa \times \omega \times Acc(\kappa) \times \omega \times \kappa \times \kappa \times \kappa \times \kappa)^{\leq\omega}$. For every $\xi \in \mathcal{T}$ there are functions $\{\xi_i \in \kappa^{\leq\omega} \mid 0 < i \leq 8\}$ such that for all $i \leq 8$, $dom(\xi_i) = dom(\xi)$ and for all $n \in dom(\xi)$, $\xi(n) = (\xi_1(n), \xi_2(n), \xi_3(n), \xi_4(n), \xi_5(n), \xi_6(n), \xi_7(n), \xi_8(n))$. For every $\xi \in \mathcal{T}$ let us denote $(\xi_4, \xi_5, \xi_6, \xi_7, \xi_8)$ by $\overline{\xi}$.

Definition 4.36. *For all $\alpha \in Acc(\kappa)$ and $\eta \in \mathcal{T}$ with $\overline{\eta} \in J_f$, $dom(\eta) = m < \omega$ define Γ^α_η as follows:*

If $\overline{\eta} \in J^\alpha_f$, then Γ^α_η is the set of elements of $\mathcal{T}$ such that:

1. *$\xi \restriction m = \eta$,*

2. *$\overline{\xi} \restriction dom(\xi) \backslash m \in W^\alpha_{\overline{\eta}}$,*

3. *ξ_3 is constant on $dom(\xi) \backslash m$,*

4. *$\xi_3(m) = \alpha$,*

5. *for all $n \in dom(\xi) \backslash m$, let $\xi_2(n)$ be the unique $r < \omega$ such that $\sigma^\alpha_\zeta(r) = \overline{\xi}(n)$, where $\zeta = \overline{\xi} \restriction n$.*

If $\overline{\eta} \notin J^\alpha_f$, then $\Gamma^\alpha_\eta = \emptyset$.

For $\eta \in \mathcal{T}$ with $\overline{\eta} \in J_f$, $dom(\eta) = m < \omega$ define

$$\Gamma(\eta) = \bigcup_{\alpha \in Acc(\kappa)} \Gamma^\alpha_\eta.$$

Finally we can define A^f by induction. Let $T_f(0) = \{\emptyset\}$ and for all $n < \omega$,

$$T_f(n+1) = T_f(n) \cup \bigcup_{\eta \in T_f(n)\ dom(\eta)=n} \Gamma(\eta),$$

for $n = \omega$,

$$T_f(\omega) = \bigcup_{n<\omega} T_f(n).$$

For $0 < i \leq 8$ let us denote by $s_i(\eta) = sup\{\eta_i(n) \mid n < \omega\}$ and $s_\omega(\eta) = sup\{s_i(\eta) \mid i \leq 8\}$, finally

$$A^f = T_f(\omega) \cup \{\eta \in \mathcal{T} \mid dom(\eta) = \omega, \forall m < \omega(\eta \restriction m \in T_f(\omega))\}.$$

Define the color function d_f by $d_f(\eta) = c_f(\overline{\eta})$ if $s_1(\eta) < s_\omega(\eta)$ and $d_f(\eta) = f(s_1(\eta))$ otherwise.

It is clear that A^f is closed under initial segments, indeed the relations $\prec$, $(P_n)_{n\leq\omega}$, and $\wedge$ of Definition 4.33 have a canonical interpretation in A^f.

Let I be the κ-colorable linear order given by Fact 4.35.

Let us proceed to define $<\restriction Suc_{A^f}(\eta)$. Let $\mathscr{H} : I \to \kappa$ be a κ-coloration of I.

For any $\eta \in A^f$ with domain m, we will define the order $<\restriction Suc_{A^f}(\eta)$ such that it is isomorphic to I. By the construction of A^f, an isomorphism between $\{(\theta_1, \theta_2, \theta_3) \in \kappa \times \omega \times Acc(\kappa) \mid \theta_3 \geq \eta_3(m-1)\}$ and I induces an order in $<\restriction Suc_{A^f}(\eta)$.

Definition 4.37. *Recall that $\mathscr{H}$ is a κ-coloration of I. For all $\theta, \alpha < \kappa$ fix the bijections $\tilde{G}_\theta : \{(\theta_2, \theta_3) \in \omega \times Acc(\kappa) \mid \theta_3 \geq \theta\} \to \kappa$ and $\tilde{H}_\alpha : \mathscr{H}^{-1}[\alpha] \to \kappa$. Notice that these functions exist because $\mathscr{H}$ is a κ-coloration of I and there are κ tuples (θ_2, θ_3).*

Let us define $\tilde{\mathcal{G}}_\theta : \{(\theta_1, \theta_2, \theta_3) \in \kappa \times \omega \times Acc(\kappa) \mid \theta_3 \geq \theta\} \to I$ by $\tilde{\mathcal{G}}_\theta((\theta_1, \theta_2, \theta_3)) = a$ where a is the unique element that satisfies:

- *$\tilde{G}_\theta((\theta_2, \theta_3)) = \alpha$;*
- *$\tilde{H}_\alpha(a) = \theta_1$.*

For any $\eta \in A^f$ with domain $m < \omega$ and $\eta_3(m-1) = \theta$, the isomorphism $\tilde{\mathcal{G}}_\theta$ induces an order in $Suc_{A^f}(\eta)$. Let us define $<\restriction Suc_{A^f}(\eta)$ as the induced order given by $\tilde{\mathcal{G}}_\theta$. It is clear that $(A^f, \prec, (P_n)_{n<\omega}, <, \wedge)$ is isomorphic to a subtree of $I^{\leq\omega}$ as in Definition 4.33.

Lemma 4.38 ([15], Theorem 3.11)**.** *Suppose I is a κ-colorable linear order. Then for all $f, g \in 2^\kappa$,*

$$f \ =^2_\omega \ g \ \Leftrightarrow A^f \cong A^g.$$

Define the tree $A_f \subseteq A^f$ by: $x \in A_f$ if and only if x is not a leaf of A^f or x is a leaf such that $d_f(x) = 1$.

Successor cardinals

We will use the generalized Ehrenfeucht-Mostowski models, see [19] Chapter VII. 2 or [10] Section 8.

Definition 4.39 (Generalized Ehrenfeucht-Mostowski models)**.** *We say that a function Φ is proper for K^γ_{tr}, if there is a vocabulary $\mathcal{L}^1$ and for each $A \in K^\gamma_{tr}$, model $\mathcal{M}_1$, and tuple a_s, $s \in A$, of elements of $\mathcal{M}_1$ the following two hold:*

- *every element of $\mathcal{M}_1$ is an interpretation of some $\mu(a_s)$, where μ is a $\mathcal{L}^1$-term;*
- *$tp_{at}(a_s, \emptyset, \mathcal{M}_1) = \Phi(tp_{at}(s, \emptyset, A))$.*

Notice that for each A, the previous conditions determine $\mathcal{M}_1$ up to isomorphism. We may assume $\mathcal{M}_1$, a_s, $s \in A$, are unique for each A. We denote $\mathcal{M}_1$ by $EM^1(A, \Phi)$. We call $EM^1(A, \Phi)$ an Ehrenfeucht-Mostowski model.

Suppose T is a countable complete theory in a countable vocabulary $\mathcal{L}$, $\mathcal{L}^1$ a Skolemization of $\mathcal{L}$, and T^1 the Skolemization of T by $\mathcal{L}^1$. If there is Φ a proper function for K^λ_{tr}, then for every $A \in K^\gamma_{tr}$, we will denote by $\mathrm{EM}(A, \Phi)$ the $\mathcal{L}$-reduction of $EM^1(A, \Phi)$. The following result ensure the existence of a proper function Φ for unsuperstable theories T and $\gamma = \omega$.

Theorem 4.40 (Shelah, [19] Theorem 1.3, proof in [19] Chapter VII 3)**.** *Suppose $\mathcal{L} \subseteq \mathcal{L}^1$ are vocabularies, T is a complete first order theory in $\mathcal{L}$, T^1 is a complete theory in $\mathcal{L}^1$ extending T and with Skolem-functions. Suppose T^1 is unsuperstable and $\{\phi_n(x, y_n) \mid n < \omega\}$ witnesses this. Then there is a function Φ proper such that for all $A \in K^\omega_{tr}$, $EM^1(A, \Phi)$ is a model of T^1, and for $s \in P^A_n$, $t \in P^A_\omega$, $EM^1(A, \Phi) \models \phi_n(a_t, a_s)$ if and only if $A \models s \prec t$.*

For every $f \in 2^\kappa$, let us denote by $\mathcal{A}^f$ the model $\mathrm{EM}(A_f, \Phi)$.

Lemma 4.41 ([15], Lemma 4.28)**.** *If T is a countable complete unsuperstable theory over a countable vocabulary, then for all $f, g \in 2^\kappa$, $f \ =^2_\omega \ g$ if and only if $\mathcal{A}^f$ and $\mathcal{A}^g$ are isomorphic.*

Theorem 4.42 ([15], Corollary 4.12)**.** *Suppose $\kappa = \lambda^+ = 2^\lambda$ and $\lambda^\omega = \lambda$. If T_1 is a countable complete classifiable theory, and T_2 is a countable complete unsuperstable theory, then $\cong_{T_1} \hookrightarrow_c \cong_{T_2}$ and $\cong_{T_2} \not\hookrightarrow_c \cong_{T_1}$.*

In [16], this construction is extended to other non-classifiable theories.

Inaccessible cardinals

For κ an inaccessible cardinal, only two results are known in ZFC. Clearly the use of diamond principles like $\mathrm{Dl}^*_S(\Pi^1_2)$ would give us the same results for unsuperstable theories.

Definition 4.43. *Let T be a stable theory. T has the orthogonal chain property (OCP), if there exist $\lambda_r(T)$-saturated models of T of power $\lambda_r(T)$, $\{\mathcal{A}_i\}_{i<\omega}$, $a \notin \cup_{i<\omega}\mathcal{A}_i$, such that $t(a, \cup_{i<\omega}\mathcal{A}_i)$ is not algebraic for every $j < \omega$, $t(a, \cup_{i<\omega}\mathcal{A}_i) \perp \mathcal{A}_j$, and for every $i \leq j$, $\mathcal{A}_i \subseteq \mathcal{A}_j$.*

Exercise 4.3. *If T has the OCP, then T is unsuperstable.*

Lemma 4.44 ([9], Corollary 5.10)**.** *Let κ be an inaccessible cardinal. Assume T is stable and has the OCP, then $=^\kappa_\omega \hookrightarrow_c \cong_T$.*

Theorem 4.45 ([9], Corollary 5.11)**.** *Let κ be an inaccessible cardinal. Assume T_1 is a classifiable theory and T_2 is a stable theory with the OCP, then $\cong_{T_1} \hookrightarrow_c \cong_{T_2}$.*

Definition 4.46. *We say that a superstable theory T has the strong dimensional order property (S-DOP) if the following holds:*
There are F^a_ω-saturated models $(M_i)_{i<3}$, $M_0 \subset M_1 \cap M_2$, such that $M_1 \downarrow_{M_0} M_2$, and for every M_3 F^a_ω-prime model over $M_1 \cup M_2$, there is a non-algebraic type $p \in S(M_3)$ orthogonal to M_1 and to M_2, such that it does not fork over $M_1 \cup M_2$.

Lemma 4.47 ([17], Corollary 5.1)**.** *Let κ be an inaccessible cardinal. Assume T is a theory with S-DOP and let λ be $(2^\omega)^+$, then $=^\kappa_\lambda \hookrightarrow_c \cong_T$.*

Theorem 4.48 ([17], Corollary 5.2)**.** *Let κ be an inaccessible cardinal. Assume T_1 is a classifiable theory and T_2 is a superstable theory with S-DOP, then $\cong_{T_1} \hookrightarrow_c \cong_{T_2}$.*

Question 4.49. *Let κ be an inaccessible cardinal, T_1 a classifiable theory, and T_2 a non-classifiable theory. Is $\cong_{T_1} \hookrightarrow_c \cong_{T_2}$ a theorem of ZFC?*

5 Questions

Question 5.1. *Is the following consistent $\Delta^1_1(\kappa) = \kappa$-Borel$^* \subsetneq \Sigma^1_1(\kappa)$?*

Question 5.2. *Is $=^\kappa_\mu \hookrightarrow_B =^2_\mu$ a theorem of ZFC?*

Question 5.3. *Let κ be an inaccessible cardinal, T_1 a classifiable theory, and T_2 a non-classifiable theory. Is $\cong_{T_1} \hookrightarrow_c \cong_{T_2}$ a theorem of ZFC?*

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