Phase transitions for Gödel incompleteness

Andreas Weiermann
Ghent University

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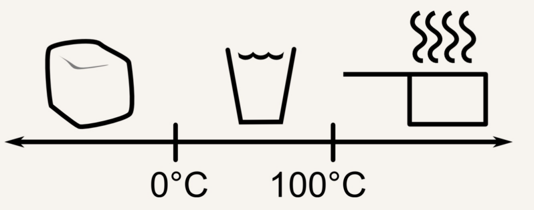
Thanks to the organizers

Aim of the talk

- Talk is of expository nature
- Addressed to non proof-theorists.
- Will keep examples simple.
- Several fairytales which are independent from each other.
- Will highlight some applications of mathematics within logic.
- Project is open ended.
- Interested students, PhD students, and Postdocs may join in.
- Search for international collaboration is intended.

Phase transition in real life



- Phase transitions in logic?
- First or second order phase transitions?
- Which methods can be used to classify them?
- Can physical methods like renormalization be useful here?

Outline

- Peano arithmetic
- Historical overview: Hilbert, Gödel,
- The big picture behind phase transitions for incompleteness
- Ordinals for beginners
- Ackermann function
- Goodstein sequences
- Ramseyan theorems
- Slowly well orderedness
- Kruskal's tree theorem
- Bolzano Weierstrass and Abel's theorem about integrals

Peano arithmetic

How to model the natural numbers?
Via Peano arithmetic PA.
Have symbols for $0, S, +$ and $\cdot$ together with the axioms:

- $\forall x[\neg Sx = 0]$.
- $\forall x, y[Sx = Sy \rightarrow x = y]$.
- $\forall x[x + 0 = x]$.
- $\forall x, y[x + Sy = S(x + y)]$.
- $\forall x[x \cdot 0 = 0]$.
- $\forall x, y[x \cdot Sy = x \cdot y + x]$.
- For all formulas φ: $(\varphi(0) \land \forall x[\varphi(x) \rightarrow \varphi(Sx)]) \rightarrow \forall y \varphi(y)$.

PA is good for developing discrete mathematics.

Compactness

Is every countable model of *PA* isomorphic to the standard model?

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Is every countable model of *PA* isomorphic to the standard model?
No, by compactness.

Inconsistency in naïve set theory

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Possible way out: Intuitionism. Predicativism.
ZFC: Replace comprehension by separation/replacement.

Hilbert: Let's save mathematics via proof theory

- Formalize all of mathematics:
  Is every assertion true in the standard model provable in PA?
- Prove the consistency of mathematics by finitary means.
  Is the consistency of PA provable in PA?

Enter Gödel

Is every assertion true in the standard model provable in *PA*?
Gödel one simplified: No
Is the consistency of *PA* provable in *PA*?
Gödel two simplified: No

Phase transitions in logic

Assumptions:

- $A(h)$ is true for all values of the order parameter h.
- $A(h)$ is provable for small values of h.
- $A(h)$ is unprovable for large enough values of h.
- Unprovability of $A(h)$ is monotone in h.

Then we are looking for the threshold for the transition from provability to unprovability.

Ordinals for beginners

Following Cantor ordinals model counting into the infinite:
$0 < 1 < 2 < \ldots < \omega < \omega + 1 < \ldots < \omega + \omega \ldots.$

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Let E be the least class of unary functions such that

- $m \mapsto 0 \in E$
- If $g, h \in E$ then $m \mapsto m^{g(m)} + h(m) \in E$.

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Then $\langle E, < \rangle$ is a well order. So $<$ is a linear ordering and there is no sequence $f_0, f_1, \ldots \in E$ such that $f_{i+1} < f_i$ for all i.

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Theorem

- *Every $\alpha \in E$ can be written uniquely as $\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$ where $\alpha > \alpha_1 \geq \ldots \geq \alpha_n$.*

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- *Every* $\alpha \in E$ *can be written uniquely as* $\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$ *where* $\alpha > \alpha_1 \geq \ldots \geq \alpha_n$.
- *If* $\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$ *where* $\alpha > \alpha_1 \geq \ldots \geq \alpha_n$ *and* $\beta = \omega^{\beta_1} + \cdots + \omega^{\beta_m}$ *where* $\beta > \beta_1 \geq \ldots \geq \beta_m$ *then* $\alpha < \beta$

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- *If* $\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$ *where* $\alpha > \alpha_1 \geq \ldots \geq \alpha_n$ *and* $\beta = \omega^{\beta_1} + \cdots + \omega^{\beta_m}$ *where* $\beta > \beta_1 \geq \ldots \geq \beta_m$ *then* $\alpha < \beta$ *iff either* $n < m$ *and* $(\forall i \leq n)[\alpha_i = \beta_i]$, *or there exists an* $i \leq \min(m, n)$ *such that* $\alpha_i < \beta_i$ *and* $(\forall j < i)[\alpha_j = \beta_j]$.

Gentzen

Assume that $\langle E, < \rangle$ is order isomorphic to $\langle \mathbb{N}, \prec \rangle$.
$I(\alpha) := \forall x[(\forall y(y \prec x \varphi(y)) \to \varphi(x)] \to \forall x \prec \alpha \varphi(x)$
$I := \forall x[(\forall y(y \prec x \varphi(y)) \to \varphi(x)] \to \forall x \varphi(x)$
Gentzen:

- PA proves $I(\alpha)$ for any $\alpha \in E$.
- PA does not prove I.

This is first example of a (meta mathematical) phase transition for Gödel incompleteness.
$PA + I$ proves the consistency of PA.

Phase transition for the Ackermann function

$$
\begin{aligned}
A_0(n) &:= n+1. \\
A_{d+1}(n) &:= A_d^n(n) = \overbrace{A_d(\ldots A_d}^{n-\text{times}}(n)). \\
A(n) &:= A_n(n).
\end{aligned}
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Then A_d is primitive recursive for any fixed d but the function A is not prim rec.

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Then A_d is primitive recursive for any fixed d but the function A is not prim rec. Introduce a parameter $h : \mathbb{N} \to \mathbb{N}$.

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\begin{aligned}
A(h)_0(n) &:= n+1. \\
A(h)_{d+1}(n) &:= A(h)_d^{h(n)}(n). \\
A(h)(n) &:= A(h)_n(n).
\end{aligned}
$$

Then for $h = id$ the function $A(h)$ is not prim rec but $A(h)$ is prim rec for $h = const$. Is there a non trivial phase transition?

Phase transition for Ackermann function

We round h up to natural number values.

Theorem

- *If* $h(n) = \sqrt[d]{n}$ *then* $A(h)$ *is not prim rec.*
- *If* $h(n) = \log(n)$ *then* $A(h)$ *is prim rec.*

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For a non decreasing unbounded function h let $h^{-1}(n) := \min\{m : h(m) \geq n\}$. Then for quickly growing functions h the induced function h^{-1} is slowly growing.

Theorem (Omri and W.)

- *If* $h(n) = \sqrt[A^{-1}(n)]{n}$ *then* $A(h)$ *is not prim rec.*
- *If* $h(n) = \sqrt[A_d^{-1}(n)]{n}$ *for some fixed* d *then* $A(h)$ *is prim rec.*

Proof idea: $A^{-1}(n)$ behaves like a constant for the branches of the Ackermann function.

Phase transition for the generalized Ackermann function

Let g be strictly increasing such that $g(n) \geq n+1$.

$$\begin{aligned} A(g,h)_0(n) &:= g(n). \\ A(g,h)_{d+1}(n) &:= A(g,h)_d^{h(n)}(n). \\ A(g,h)(n) &:= A(g,h)_n(n). \end{aligned}$$

A classification result for a general function g seems out of reach

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A classification result for a general function g seems out of reach but our search was successful. Let $g'(n) := g^n(n)$.

Theorem

- *If* $h(n) = \sqrt[A^{-1}(n)]{(g')^{-1}(n)}$ *then* $A(g,h)$ *is not prim rec.*
- *If* $h(n) = \sqrt[A_d^{-1}(n)]{(g')^{-1}(n)}$ *for some fixed* d *then* $A(g,h)$ *is prim rec.*

This theorem has applications to the pigeonhole principle. Moreover this theorem extends to transfinite extensions $A_\alpha(g,h)$ but we will skip further details here.

Phase transition for the Goodstein principle

Fix a natural number $k \geq 2$.

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Fix a natural number $k \geq 2$. Every positive integer m can be written uniquely as $m = k^r \cdot p + q$ where $k^r \leq m < k^{r+1}$ and $p < k$ and $q < k^r$.

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These numbers grow quickly in the beginning and every laptop will make you believe that m_l is diverging for not too small starting values of m.
Example: $m = 2^2 + 2$. $m_0 = m$. $m_1 = 3^3 + 3 - 1 = 3^3 + 2$. etc.

Theorem

Let G stand for $\forall m \exists l (m_l = 0)$.

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Theorem

Let G stand for $\forall m \exists l (m_l = 0)$.

- *The assertion G is true.*
- *But $PA \nvdash G$.*

Mentioned by Penrose.

Phase transition for the Goodstein principle

Let us introduce a function parameter h for G where $h : \mathbb{N} \to \mathbb{N}$ is non decreasing.
Every positive integer m can be written uniquely as $m = (2 + h(k))^r \cdot p + q$ where $(2 + h(k))^r \leq m < (2 + h(k))^{r+1}$ and $p < 2 + h(k)$ and $q < (2 + h(k))^r$. Define $0\{k\}^h := 0$ and $m\{k\}^k := (2 + h(k + 1))^{r\{k\}^h} \cdot p + q\{k\}^h$. Define $m_0^h := m$ and if $m_l^h > 0$ let $m_{l+1}^h := m_l^h\{l\}^h - 1$. If $m_l = 0$ then $m_{l+1}^h := 0$.

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Theorem

Let $G(h)$ stand for $\forall m \exists l (m_l^h = 0)$. Let $\log$ stand for the binary length function. $\log^1 := \log$ and $\log^{d+1} := \log(\log^d)$. Let $\log^*(m)$ be the least number d such that $\log^d(m) \leq 1$.

- *The assertion $G(h)$ is true for all h.*
- *$PA \vdash G(\log^*)$.*
- *But for any fixed d $PA \nvdash G(\log^d)$.*

Phase transition for the Paris Harrington principle

We identify R with the set $\{0, 1, \ldots, R-1\}$. With $[X]^d$ we denote the set of d-element subsets of X.
The finite Ramsey theorem RT states:
$\forall d, c, m \, \exists R \forall P : [R]^d \to c \exists H \subseteq R[P \restriction [H]^d = const \wedge |H| \geq m]$.
The least R depending on d, c, m is denoted by $R_c^d(m)$.

Theorem

- *RT is true.*
- *$PA \vdash RT$.*
- *There exists a constant const such that* $R_c^d(m) \leq 2_{d-1}(const \cdot c \cdot m)$.

No phase transition yet.

Phase transition for the Paris Harrington principle

The Paris Harrington thm PH states: $\forall d, c, m \exists R \forall P : [R]^d \to c$ $\exists H \subseteq R[P \restriction [H]^d = const \wedge |H| \geq \max\{\min(H), m\}]$.

Theorem (Paris Harrington)

- *PH is true.*
- *$PA \nvdash PH$.*

This indicates that a phase transition might be possible.

Phase transition for the Paris Harrington principle

The Paris Harrington theorem $PH(h)$ states: $\forall d, c, m \exists R \forall P :$ $[R]^d \to c \exists H \subseteq R[P \restriction [H]^d = const \wedge |H| \geq \max\{h(\min(H)), m\}]$.

Theorem (W.)

- *$PH(h)$ is true.*
- *$PA \vdash PH(\log^*)$.*
- *If d is fixed then $PA \nvdash PH(\log^d)$.*

Similar results hold for the regressive Ramsey theorem and the canonical Ramsey theorem. Threshold is similar to the one for Goodstein. Further refinements are possible.

Phase transition for the Paris Harrington principle

Proof of $PA \vdash PH(\log^*)$.
Let m, d, c be given and put $R := R_c^{d\cdot 2}(m \cdot 2)$. W.l.o.g $m = d = c$. Let $P : [R]^d \to c$ be given. Then by RT there exists H such that $P \restriction [H]^d = const$ and $2m \leq |H|$. But now we have the following insightful conclusion using the Erdös Rado bound
$\log^*(\min H) \leq \log^*(R) \leq \log^*(2_{2m}) = 2m \leq |H|$.

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Insight: Finite combinatorics may indicate the threshold function. The threshold emerges as the functional inverse of the bounding function.

Ordinal counting

Let us consider ordinals in E. Let $N\alpha := n + N\alpha_1 + \cdots + N\alpha_n$ if $\alpha = \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$ where $\alpha > \alpha_1 \geq \ldots \geq \alpha_n$.
Then for all d the set of all $\alpha \in E$ with $N\alpha \leq d$ is finite (bounded by an exponential upper bound in n).
Let $c_\alpha(n) = \#\{\beta < \alpha : N\beta = n\}$.
Then $c_{\omega^d}(n) \sim const \cdot n^{d-1}$.
What about other values of $c_\alpha(n)$?

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Theorem

Let $\omega_1 = \omega$ and $\omega_{d+1} = \omega^{\omega_d}$.

- $c_{\omega^\omega}(n) \sim$

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Theorem

Let $\omega_1 = \omega$ and $\omega_{d+1} = \omega^{\omega_d}$.

- $c_{\omega^\omega}(n) \sim \frac{e^{\pi\sqrt{\frac{2}{3}\cdot n}}}{4\sqrt{3}n}$ *following Hardy and Ramanujan.*

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- $\ln(c_{\omega_{d+2}}(n)) \sim$

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- $c_{\omega^\omega}(n) \sim \frac{e^{\pi\sqrt{\frac{2}{3}\cdot n}}}{4\sqrt{3}n}$ *following Hardy and Ramanujan.*
- $\ln(c_{\omega_{d+2}}(n)) \sim \frac{\pi^2}{6} \cdot \frac{n}{\ln^d(n)}$.

Proof by additive number theory, generating functions (Odlycko).
Spin off to zero one laws for ordinals.

Slowly well orderedness a la Friedman

Let *SWO* be the following assertion.
$\forall K \exists M \forall \alpha_1, \ldots, \alpha_M [(\forall i \leq M (N\alpha_i \leq K + i)) \rightarrow \exists i < M (\alpha_i \leq \alpha_{i+1})]$.

Theorem (Friedman)

- *SWO is true.*
- $PA \nvdash SWO$.

Slowly well orderedness a la Friedman

Let $SWO(h)$ be the following assertion.
$\forall K \exists M \forall \alpha_1, \ldots, \alpha_M [(\forall i \leq M (N\alpha_i \leq K + h(i))) \rightarrow \exists i < M (\alpha_i \leq \alpha_{i+1})]$.

Theorem (W.)

- $PA \vdash SWO(\log \cdot \log^*)$.
- $PA \nvdash SWO(\log \cdot \log^d)$.

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Theorem (W.)

- $PA \vdash SWO(\log \cdot \log^*)$.
- $PA \nvdash SWO(\log \cdot \log^d)$.

Proof: $f = \log \cdot \log^d$ is the inverse of the counting function $c_{\omega_{d+2}}$ which yields the provability part by counting. For the unprovability part we apply "renormalization" to Friedman's result for $f = id$.

Slowly well orderedness a la Friedman

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Theorem (W.)

- $PA \vdash SWO(\log \cdot \log^*)$.
- $PA \nvdash SWO(\log \cdot \log^d)$.

Proof: $f = \log \cdot \log^d$ is the inverse of the counting function $c_{\omega_{d+2}}$ which yields the provability part by counting. For the unprovability part we apply "renormalization" to Friedman's result for $f = id$.

We can also vary the count function. If $N\alpha$ is given by a canonical Gödel numbering we can apply multiplicative number theory (Dirichlet generating functions) and Tauberian theory.

Kruskal's theorem

A finite tree is a finite partial order $\langle T, \leq_T \rangle$ such that

- T has a minimum r such that $\forall t \in T (r \leq_T t)$ and
- such that for all $t \in T$ the set $\{s \in T : s \leq t\}$ is totally (linearly) ordered.

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Then for $s, t \in T$ there exists $\inf(s, t) \in T$.
A one to one function $e : T \to T'$ is called a homeomorphic embedding if for all $s, t \in T$ we have
$e(\inf_T(s, t)) = \inf_{T'}(e(s), e(t))$. We say $T \trianglelefteq T'$ if there exists a homeomorphic embedding of T into T'.

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Theorem (Kruskal's theorem KT)

For every ω-sequence of finite trees T_i there exists $i, j < \omega$ such that $i < j$ and $T_i \trianglelefteq T_j$.

Friedman showed that KT is not provable in ATR_0.

Friedman's miniaturization of Kruskal's theorem FKT

Let $N(T)$ denote the number of nodes in T. Let FKT stand for:
$\forall K \exists M \forall T_1, \ldots, T_M[(\forall i \leq M(N(T_i) \leq K + i)) \rightarrow \exists i < j < M T_i \trianglelefteq T_j]$.

Theorem (Friedman)

- *FKT is true*
- *PA* $\nvdash$ *FKT. Even stronger* $ATR_0 \nvdash FKT$

Friedman's miniaturization of Kruskal's theorem FKT

Let $h : \mathbb{N} \to \mathbb{N}$. The assertion $FKT(h)$ stands for: $\forall K \exists M \forall T_1, \ldots, T_M \big[(\forall i \leq M)[(N(T_i) \leq K + h(i)] \to (\exists i, j)[i < j < M \wedge T_i \trianglelefteq T_j]\big]$.

Theorem

- *$FKT(h)$ is true.*
- *If $h = const$ then $PA \vdash FKT(h)$.*
- *$PA \nvdash FKT(id)$.*

Theorem (Matousek and Loebl)

Let $h_r(i) := r \cdot \log_2(i)$.

- *If $r \leq 0.5$ then $PA \vdash FKT(h_r)$.*
- *If $r \geq 4$ then $PA \nvdash FKT(h_r)$.*

Any guesses for the threshold?

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What is c and how did I find it?

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What is c and how did I find it?
With a search machine I looked around 2000 for "finite tree". I found Otter's constant $\alpha = 2.955765\ldots$ on Stephen Finch's pages about mathematical constants. Voila: $c = \frac{1}{\log_2(\alpha)}$.

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$t_n = \#\{T : N(T) = n\}$.
The threshold function h_c is morally the same as the inverse function of $n \mapsto t_n$. The concrete digits of c after 5 entries behind the . have been calculated by Moritz Firsching.

The scaling window for FKT

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\begin{aligned}
A_0(x) &:= x+1. \\
A_{\alpha+1}(x) &:= A_\alpha^x(x). \\
A_\lambda(x) &:= A_{\lambda[x]}(x) \text{ if } \lambda \text{ is of limit type }. \\
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The functions A_α are quickly growing.

Let $h^\alpha(i) = (c + 1/A_\alpha^{-1}(i)) \cdot \log_2(i)$ and $h^E(i) = (c + 1/A_E^{-1}(i)) \cdot \log_2(i)$. Using Goh and Schmutz (Random struct. and alg.) Gordeev and I could show the following result.

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Theorem

- *If $\alpha \in E$ then $PA \vdash FKT(h^\alpha)$.*
- *$PA \nvdash FKT(h^E)$.*

Theorem (Bolzano Weierstraß)

Let $\langle x_i : i < \omega \rangle$ be an infinite sequence of real numbers from the closed unit interval $[0, 1]$. Then there exist $k_1 < k_2 < \ldots$ such that the subsequence $\langle x_{k_i} : i < \omega \rangle$ converges.

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Theorem (Monotone convergence theorem)

*Let $\langle x_i : i < \omega \rangle$ be a non decreasing infinite sequence of real numbers from the closed unit interval $[0,1]$.
Then the sequence $\langle x_i : i < \omega \rangle$ converges.*

Friedman's BW

Theorem (modified Bolzano Weierstraß (Friedman))

Let $f : \omega \to \mathbb{R}_{>0}$. *Let* $\langle x_i : i < \omega \rangle$ *be an infinite sequence from the closed unit interval* $[0, 1]$.

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- $\int_{exp^d(1)}^{\infty} \frac{1}{x \cdot \log(x) \cdot \ldots \cdot (\log^d(x))^{1+\varepsilon}} dx < +\infty$ *for* $\varepsilon > 0$.

Friedman's miniaturization of BW

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Theorem (Friedman)

For all $K \geq 3$ there exists M such for all sequences $x_1, x_2, \ldots, x_M$ from the closed unit interval $[0, 1]$ there exist $k_1 < \ldots < k_K \leq M$ such that $|x_{k_{i+1}} - x_{k_{i+2}}| < 1/(k_i)^2$ holds for all $i \leq K - 2$.

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Let $f : \omega \to \mathbb{R}_{>0}$*. The following assertion* $FBW(f)$ *is true. For all* $K \geq 3$ *there exists* M *such for all sequences* $x_1, x_2, \ldots, x_M$ *from the closed unit interval* $[0, 1]$ *there exist* $k_1 < \ldots < k_K \leq M$ *such that* $|x_{k_{i+1}} - x_{k_{i+2}}| < f(k_i)$ *holds for all* $i \leq K - 2$.

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Friedman's Proof:
Apply the compactness of the Hilbert cube $[0,1]^{\mathbb{N}}$ (Tychonov).

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Let $f(i) = \frac{1}{i^2}$. For $K \geq 10$, the least $M(K)$ for $FBW(f)$ exceeds $Ack_{K-8}(64)$.

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What happens when we modify the estimate $1/(k_i)^2$? Obviously any strictly positive function of k_i will make these statements hold, but the issue is the size of the associated constants $M(K)$.
If we we use the estimate $\frac{1}{(k_i)^{1+\varepsilon}}$, where $\varepsilon > 0$, then (following Friedman unpublished) we obtain roughly the same Ackermannian upper and lower bounds.

Friedman's miniaturization of BW

Theorem (Friedman)

Let $f(i) := \log(i)/i$. *Then the least* $M(K)$ *for* $FBW(f)$ *is bounded by an exponential function.*

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Friedman's challenge from 1999: How does $M(K)$ behave for $f(i) = \frac{1}{i}$ and $f(i) = \frac{1}{i \log(i)}$?

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Question: Can we improve on the Ackermannian upper bounds for $f(i) = \frac{1}{i}$ and $f(i) = \frac{1}{i\log(i)}$?

Friedman's monotone convergence principle

Assume that f is strictly positive. Let $FMC(f)$ be the following assertion: For all $K \geq 3$ there exists M such for all weakly increasing sequences $x_1 \leq x_2 \leq \ldots \leq x_M$ from the closed unit interval $[0, 1]$ there exist $k_1 < \ldots < k_K \leq M$ such that $|x_{k_{i+1}} - x_{k_{i+2}}| < f(k_i)$ holds for all $i \leq K - 2$.

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- *Let* $f(i) = \frac{1}{i(\log(i))^{1+\varepsilon}}$. *Then the least* $M(K)$ *in* $FMC(f)$ *is Ackermannian.*

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Theorem (W.)

Fix $d < \omega$. Let $\log^d(i)$ denote the d-th iterate of log.

- *Let $f(i) = \frac{1}{i \cdot \log(i) \cdot \ldots \cdot \log^d(i)}$. Then the least $M(K)$ in $FMC(f)$ is bounded by a $d+1$-times iterated exponential function.*

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- *Let $f_d(i) = A_d^{-1}(i) - A_d^{-1}(i-1)$ and where A_d refers to the d-th branch of the Ackermann function, and $^{-1}$ refers to taking functional inverses.*

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Hint for the proof of the first two assertions: Do the counting argument for $FMC(f)$ but now with nested layers.

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Conjecture

$A_d^{-1}(l) - A_d^{-1}(l-1)$ will be of order $D_d(l)$ as l tends to infinity. This expression will approximate the derivative of A_d^{-1}.

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Also the phase transition for Abel's theorem could be refined by looking at a suitably defined D_α.

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