Spectra of computable models of strongly minimal disintegrated theories in rank 1 languages

Steffen Lempp, University of Wisconsin-Madison

http://www.math.wisc.edu/∼lempp

(Joint work with Uri Andrews)

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Hrushovski disproved Zil'ber's Conjecture using so-called *Hrushovski constructions* (1991) and *Hrushovski fusions* (1992).

The following theorem will allow us to define spectra:

Theorem (Baldwin/Lachlan 1971)

The countable models of any $\aleph_1$-categorical but not totally categorical theory T in any countable language form an elementary chain

$$\mathcal{M}_0 \prec \mathcal{M}_1 \prec \ldots \prec \mathcal{M}_\omega$$

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The *spectrum of computable models* of an $\aleph_1$-categorical but not totally categorical theory T in any computable language is

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Warning: $\mathcal{M}_\alpha$ may have dimension $k + \alpha$ for fixed $k > 0$.

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Corollary

For any strongly minimal disintegrated theory T, the spectrum of T is a Σ^0_5-set.

Theorem

The following are all previously known spectra of computable models of strongly minimal (indeed, all $\aleph_1$-categorical) theories:

- $\emptyset$ and $[0, \omega]$ (trivial)
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If T is a strongly minimal disintegrated theory in a *finite* language $\mathcal{L}$, then the possible spectra of computable models are exactly $\emptyset$, $[0, \omega]$, and $\{0\}$.

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In addition to disintegrated theories, the result of Andrews/Medvedev also extends to locally modular expansions of a group and, by Poizat (1988), to field-like theories, i.e., to "most" trichotomous theories.

For infinite languages, the situation is more difficult.

Theorem (Andrews/Lempp)

If T is a strongly minimal disintegrated theory in a (possibly infinite) *binary relational* language $\mathcal{L}$, then the possible spectra of computable models are exactly the following seven sets:
$\emptyset$, $[0, \omega]$, $\{0\}$, $\{1\}$, $\{0, 1\}$, $\{\omega\}$, and $[1, \omega]$.

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Our recent work has been motivated by the following sweeping

Conjecture

If T is a strongly minimal disintegrated theory in a (possibly infinite) relational language $\mathcal{L}$ of arity at most n, then there are only finitely many possible spectra of computable models.

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The following constitutes progress toward, and is related to, this conjecture.

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In a strongly minimal model $\mathcal{M}$, a relation $R \subseteq M^n$

- *has (Morley) rank* 0 if R is finite (and nonempty);
- *has (Morley) rank at most* 1 if for any $\overline{a} \in M^n$ with $\mathcal{M} \models R(\overline{a})$, $\dim(\mathrm{acl}(\overline{a}))$ is at most 1, i.e., $\overline{a}$ does not contain two mutually generic elements.

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Theorem (Andrews/Lempp)

If T is a strongly minimal disintegrated theory in a relational language $\mathcal{L}$ *of bounded arity* such that in each model $\mathcal{M}$ of T, any relation $R^{\mathcal{M}}$ has rank at most 1, then the possible spectra of computable models are among the following ten sets, of which the first seven are indeed spectra, even in binary languages:
$\emptyset$, $[0,\omega]$, $\{0\}$, $\{1\}$, $\{0,1\}$, $\{\omega\}$, $[1,\omega]$, and possibly $\{0,\omega\}$, $\{0,1,\omega\}$ and $\{1,\omega\}$.

The assumption of bounded arity in the previous theorem was crucial since we also have:

Theorem (Andrews/Lempp)

If T is a strongly minimal disintegrated theory in a relational language $\mathcal{L}$ (of any arity) such that in each model $\mathcal{M}$ of T, any relation $R^{\mathcal{M}}$ has rank at most 1, then the possible spectra of computable models are: $\emptyset$, $[0, \omega]$, $\{1\}$, $\{\omega\}$, $[1, \omega]$, $[0, n]$ for $n \in \omega$, $[0, \omega)$, and possibly $[0, n] \cup \{\omega\}$ for $n \in \omega$ as well as $\{1, \omega\}$.

With a trick, we can "almost" reduce the ternary case to the rank-1 case and obtain the following

Theorem (Andrews/Lempp)

If T is a strongly minimal disintegrated theory in a ternary relational language $\mathcal{L}$, then there are at least nine and at most eighteen possible spectra of computable models:
For any spectrum S, $[3,\omega) \cap S \neq \emptyset$ implies $[1,\omega] \subseteq S$.

Step 1: Reduce to rank 1:

Binary $\mathcal{L}$: If $\mathcal{M}_\alpha$ for some $\alpha \geq 2$ is computable, then fix two mutually generic $a, b \in M_\alpha$.

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Then $\mathcal{M}_\alpha \models \exists^\infty w\, R(w, y, z)$ iff at least two of $\mathcal{M}_\alpha \models R(a, y, z)$, $\mathcal{M}_\alpha \models R(b, y, z)$, and $\mathcal{M}_\alpha \models R(c, y, z)$ hold,

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Now all of $\exists^\infty w\, R(w, y, z)$, $\exists^\infty w\, R(x, w, z)$, $\exists^\infty w\, R(x, y, w)$, $R(x, y, z) \smallsetminus [\exists^\infty w\, R(w, y, z) \vee \exists^\infty w\, R(x, w, z) \vee \exists^\infty w\, R(x, y, w)]$, $[\exists^\infty w\, R(w, y, z) \vee \exists^\infty w\, R(x, w, z) \vee \exists^\infty w\, R(x, y, w)] \smallsetminus R(x, y, z)$ have rank at most 1 and are effectively interdefinable with $R(x, y, z)$.

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For a basis B of a strongly minimal disintegrated model $\mathcal{M}_\alpha$, we have

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Suppose

- $\mathcal{M}_\beta \subset \mathcal{M}_\alpha$ for $\beta < \alpha \leq \omega$,
- $\mathcal{M}_\alpha$ is a computable model,
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Then $\mathcal{M}_\beta$ has a computable copy:
Let $\dim(\mathcal{M}_\beta) = k + \beta$, fix $k + \beta$ many mutually generics $\overline{a}$ in M_β and construct $\mathrm{acl}(\overline{a})$, "discarding mistakes" into S.

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Proof:

Define the n-neighborhood $\mathrm{Nbh}_n(a)$ of $a \in M_\alpha$ by recursion:

$$\mathrm{Nbh}_0(a) = \{a\}$$

$$\mathrm{Nbh}_{n+1}(a) = \{b \in M_\alpha \mid \exists c \in \mathrm{Nbh}_n(a)\,[c, b \text{ “directly connected”}]\}$$

where c and b are "directly connected" if the binary projection of an m-ary relation $R \in \mathcal{L}$ holds (or fails) between c and b but not between c and cofinitely many elements of M_α, nor between b and cofinitely many elements of M_α.

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Then $\mathbf{0}'$ can compute canonical indices for $\mathrm{Nbh}_n(a)$ (uniformly in n but *non*uniformly in a).

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Assume $\mathcal{L}$ is "closed under permutation of variables".
Define the set of "bad elements"

$$B = \{b \in M_k \mid \exists i \, \exists^\infty y \, \exists \overline{z} \, R_i(b, y, \overline{z})\}$$

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Case II: B is infinite: Then acl($\emptyset$) contains an infinite Σ^0_1-subset B in $\mathcal{M}_k$.

In either case, we can apply the previous steps to see that $\mathcal{M}_{k-1}$ is computable.

Step 5: "Up": If all relations in $\mathcal{M}_\alpha \models T$ are at most rank 1 and of bounded arity, and if $k \in \mathrm{SCM}(T) \cap [2, \omega)$, then $k + 1 \in \mathrm{SCM}(T)$ (uniformly in k; so $\omega \in \mathrm{SCM}(T)$ as well):

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Again, assume $\mathcal{L}$ is "closed under permutation of variables".

Case I: For generic $a \in M_k$, there are infinitely many disjoint tuples $\overline{b}$ in M_k such that

$$\mathcal{M}_k \models \exists i \left(R_i(a, \overline{b}) \wedge \exists^{<\infty} x \, R_i(x, \overline{b}) \right)$$

Step 5: "Up": If all relations in $\mathcal{M}_\alpha \models T$ are at most rank 1 and of bounded arity, and if $k \in \mathrm{SCM}(T) \cap [2, \omega)$, then $k + 1 \in \mathrm{SCM}(T)$ (uniformly in k; so $\omega \in \mathrm{SCM}(T)$ as well):

Again, assume $\mathcal{L}$ is "closed under permutation of variables".

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Then we can generate a Σ^0_1-set of such disjoint tuples and then construct $\mathcal{M}_{k+1}$ as $\mathcal{M}_k \sqcup \mathrm{iacl}(g)$ for a new generic element g.

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Case II: Otherwise there is a finite set $\{h_0, \ldots, h_n\}$ of elements involved in all R_i: We can then generate a new language $\mathcal{L}'$ of *lower* arity consisting of all R_i with fixed h_j, and iterate Case I vs. Case II for $\mathcal{L}'$, etc., until we reach Case I or a binary language.

Binary $\mathcal{L}$: We also need to show

$$\{0,1\} \cap \mathrm{SCM}(T) \neq \emptyset \text{ and } \omega \in \mathrm{SCM}(T) \implies 2 \in \mathrm{SCM}(T)$$

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Finally: Several priority arguments to establish new spectra.

Thanks!