Phase transition for the late points of random walk

Let X be a simple random walk in $\mathbb{Z}_n^d$ with $d \geq 3$ and let t_{cov} be the expected time it takes for X to visit all vertices of the torus. In joint work with Prévost and Rodriguez we study the set $\mathcal{L}_\alpha$ of points that have not been visited by time αt_{cov} and prove that it exhibits a phase transition: there exists α_* so that for all $\alpha > \alpha_*$ and all $\epsilon > 0$ there exists a coupling between $\mathcal{L}_\alpha$ and two i.i.d. Bernoulli sets $\mathcal{B}^\pm$ on the torus with parameters $n^{-(a \pm \epsilon)d}$with the property that $\mathcal{B}^- \subseteq \mathcal{L}_\alpha \subseteq \mathcal{B}^+$ with probability tending to 1 as $n \to \infty$. When $\alpha \leq \alpha_*$, we prove that there is no such coupling.