Forcing techniques for Cichoń's Maximum

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Abstract

Cichoń's diagram describes the connections between combinatorial notions related to measure, category, and compactness of sets of irrational numbers. In the second part of the 2010's decade, Goldstern, Kellner and Shelah constructed a forcing model of Cichoń's Maximum (meaning that all non-dependent cardinal characteristics are pairwise different) by using large cardinals. Some years later, we eliminated this large cardinal assumption. In this mini-course, we explore the forcing techniques to construct the Cichoń's Maximum model and much more.

1 Tukey connections and cardinal characteristics of the continuum

Great part of the contents of this section are taken almost verbatim from: Section 1, up to Figure 3, of [CM22]; and Section 1, up to Fact 1.2, of [GKMS21b].

Many cardinal characteristics of the continuum and their relations can be represented by relational systems as follows. This presentation is based on [Voj93, Bar10, Bla10].

Definition 1.1. We say that $R = \langle X, Y, \sqsubset \rangle$ is a *relational system* if it consists of two non-empty sets X and Y and a relation $\sqsubset$.

(1) A set $F \subseteq X$ is *R-bounded* if $\exists\, y \in Y\ \forall\, x \in F \colon x \sqsubset y$.

(2) A set $E \subseteq Y$ is *R-dominating* if $\forall\, x \in X\ \exists\, y \in E \colon x \sqsubset y$.

We associate two cardinal characteristics with this relational system R:

$\mathfrak{b}(R) := \min\{|F| \colon F \subseteq X \text{ is } R\text{-unbounded}\}$ the *unbounding number of R*, and

$\mathfrak{d}(R) := \min\{|D| \colon D \subseteq Y \text{ is } R\text{-dominating}\}$ the *dominating number of R*.

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Note that $\mathfrak{d}(R) = 1$ iff $\mathfrak{b}(R)$ is undefined (i.e. there are no R-unbounded sets, which is the same as saying that X is R-bounded). Dually, $\mathfrak{b}(R) = 1$ iff $\mathfrak{d}(R)$ is undefined (i.e. there are no R-dominating families).

A very representative general example of relational systems is given by directed preorders.

Definition 1.2. We say that $\langle S, \leq_S \rangle$ is a *directed preorder* if it is a preorder (i.e. $\leq_S$ is a reflexive and transitive relation on S) such that

$$\forall\, x, y \in S\ \exists\, z \in S\colon x \leq_S z \text{ and } y \leq_S z.$$

A directed preorder $\langle S, \leq_S \rangle$ is seen as the relational system $S = \langle S, S, \leq_S \rangle$, and their associated cardinal characteristics are denoted by $\mathfrak{b}(S)$ and $\mathfrak{d}(S)$. The cardinal $\mathfrak{d}(S)$ is actually the *cofinality of* S, typically denoted by $\mathrm{cof}(S)$ or $\mathrm{cf}(S)$.

Fact 1.3. *If a directed preorder S has no maximum element then $\mathfrak{b}(S)$ is infinite and regular, and $\mathfrak{b}(S) \leq \mathrm{cf}(\mathfrak{d}(S)) \leq \mathfrak{d}(S) \leq |S|$. Even more, if L is a linear order without maximum then $\mathfrak{b}(L) = \mathfrak{d}(L) = \mathrm{cof}(L)$.*

Proof. First notice that $\mathfrak{d}(S)$ is infinite, otherwise, by directedness, $\mathfrak{d}(S) = 1$ and S would have a top element.

We prove the less obvious $\mathfrak{b}(S) \leq \mathrm{cf}(\mathfrak{d}(S))$. Assume that $\lambda < \mathfrak{b}(S)$ is a cardinal and $\langle A_\alpha \colon \alpha < \lambda \rangle$ is a sequence of subsets of S of size $<\mathfrak{d}(S)$. It is enough to show that $A := \bigcup_{\alpha<\lambda} A_\alpha$ is not cofinal in S. For each $\alpha < \lambda$, since $|A_\alpha| < \mathfrak{d}(S)$, A_α is not cofinal in S, so there is some $x_\alpha \in S$ such that $x_\alpha \not\leq_S y$ for all $y \in A_\alpha$. Now, $|\{x_\alpha \colon \alpha < \lambda\}| \leq \lambda < \mathfrak{b}(S)$, so there is some $x^* \in S$ such that $x_\alpha \leq_S x^*$ for all $\alpha < \lambda$. Then, $x^* \not\leq_S y$ for all $y \in A$, i.e. A is not cofinal in S.

A similar argument shows that $\mathfrak{b}(S)$ is regular. □

Example 1.4. Consider $\omega^\omega = \langle \omega^\omega, \leq^* \rangle$, which is a directed preorder. The cardinal characteristics $\mathfrak{b} := \mathfrak{b}(\omega^\omega)$ and $\mathfrak{d} := \mathfrak{d}(\omega^\omega)$ are the well-known *bounding number* and *dominating number*, respectively.

Example 1.5. For any ideal $\mathcal{I}$ on X, we consider the following relational systems.

(1) $\mathcal{I} := \langle \mathcal{I}, \subseteq \rangle$ is a directed partial order. Note that

$$\mathfrak{b}(\mathcal{I}) = \mathrm{add}(\mathcal{I}) := \min\left\{|\mathcal{F}| \colon \mathcal{F} \subseteq \mathcal{I},\ \bigcup \mathcal{F} \notin \mathcal{I}\right\} \text{ the } \textit{additivity of } \mathcal{I},$$
$$\mathfrak{d}(\mathcal{I}) = \mathrm{cof}(\mathcal{I}).$$

(2) $\mathbf{C}_\mathcal{I} := \langle X, \mathcal{I}, \in \rangle$. When $\bigcup \mathcal{I} = X$,

$$\mathfrak{b}(\mathbf{C}_\mathcal{I}) = \mathrm{non}(\mathcal{I}) = \min\{|F| \colon F \subseteq X,\ F \notin \mathcal{I}\} \text{ the } \textit{uniformity of } \mathcal{I},$$
$$\mathfrak{d}(\mathbf{C}_\mathcal{I}) = \mathrm{cov}(\mathcal{I}) = \min\left\{|\mathcal{C}| \colon \mathcal{C} \subseteq \mathcal{I},\ \bigcup \mathcal{C} = X\right\} \text{ the } \textit{covering of } \mathcal{I}.$$

Definition 1.6. Let $\Xi \colon \mathbb{B} \to [0, \infty]$ be a fam (finitely additive measure) on a Boolean algebra $\mathbb{B}$. We define the Ξ-*null ideal* by

$$\mathcal{N}(\Xi) := \{a \in \mathbb{B} \colon \Xi(a) = 0\}.$$

When $\mathbb{B}$ is a field of sets over X, we extend the definition to

$$\mathcal{N}(\Xi) := \{a \subseteq X : \exists b \in \mathbb{B} : a \subseteq b \text{ and } \Xi(b) = 0\}.$$

This is clearly an ideal on X. When $\bigcup \mathcal{N}(\Xi) = X$, i.e. every singleton has measure zero, we say that the fam Ξ is *free*.

Denote by Lb the Lebesgue measure on $\mathbb{R}$, and let $\mathcal{N} := \mathcal{N}(\mathrm{Lb})$.

Definition 1.7. Let X be a topological space. We say that $F \subseteq X$ is *nowhere dense (nwd)* if, for any non-empty open $U \subseteq X$, there is some non-empty open $U' \subseteq U$ disjoint from F. We say that $A \subseteq X$ is *meager (or of first category)* if $A = \bigcup_{n<\omega} F_n$ for some nwd F_n $(n < \omega)$.

Denote by $\mathcal{M}(X)$ the collection of all meager subsets of X, and let $\mathcal{M} := \mathcal{M}(\mathbb{R})$.

Definition 1.8. Define by $\mathcal{E}$ the ideal generated by the F_σ measure zero subsets of $\mathbb{R}$.

It is clear that $\mathcal{E} \subseteq \mathcal{M} \cap \mathcal{N}$, even more, $\mathcal{E} \subsetneq \mathcal{M} \cap \mathcal{N}$ ([BJ95, Lem. 2.6.1], see also [GM23, Thm. 3.7]).

Example 1.9. Define the relational system $\mathrm{Spl} := \langle [\omega]^{\aleph_0}, [\omega]^{\aleph_0}, \sqsubset^{\mathrm{nsp}} \rangle$ by

$$a \sqsubset^{\mathrm{nsp}} b \text{ iff either } a \supseteq^* b \text{ or } \omega \smallsetminus a \supseteq^* b.$$

Note that $a \not\sqsubset^{\mathrm{nsp}} b$ iff a splits b, so $\mathfrak{b}(\mathrm{Spl}) = \mathfrak{s}$ and $\mathfrak{d}(\mathrm{Spl}) = \mathfrak{r}$, the *splitting* and *reaping* numbers, respectively.

Inequalities between cardinal characteristics associated with relational systems can be determined by the dual of a relational system and also via Tukey connections, which we introduce below.

Definition 1.10. If $R = \langle X, Y, \sqsubset \rangle$ is a relational system, then its *dual relational system* is defined by $R^\perp := \langle Y, X, \sqsubset^\perp \rangle$ where $y \sqsubset^\perp x$ if $\neg(x \sqsubset y)$.

Fact 1.11. *Let $R = \langle X, Y, \sqsubset \rangle$ be a relational system.*

(a) $(R^\perp)^\perp = R$.

(b) The notions of $R^\perp$-dominating set and R-unbounded set are equivalent.

(c) The notions of $R^\perp$-unbounded set and R-dominating set are equivalent.

(d) $\mathfrak{d}(R^\perp) = \mathfrak{b}(R)$ *and* $\mathfrak{b}(R^\perp) = \mathfrak{d}(R)$.

Definition 1.12. Let $R = \langle X, Y, \sqsubset \rangle$ and $R' = \langle X', Y', \sqsubset' \rangle$ be relational systems. We say that $(\Psi_-, \Psi_+) : R \to R'$ is a *Tukey connection from R into R'* if $\Psi_- : X \to X'$ and $\Psi_+ : Y' \to Y$ are functions such that

$$\forall x \in X\ \forall y' \in Y' : \Psi_-(x) \sqsubset' y' \Rightarrow x \sqsubset \Psi_+(y').$$

The *Tukey order* between relational systems is defined by $R \preceq_{\mathrm{T}} R'$ iff there is a Tukey connection from R into R'. *Tukey equivalence* is defined by $R \cong_{\mathrm{T}} R'$ iff $R \preceq_{\mathrm{T}} R'$ and $R' \preceq_{\mathrm{T}} R$

Fact 1.13. *Assume that $R = \langle X, Y, \sqsubset\rangle$ and $R' = \langle X', Y', \sqsubset'\rangle$ are relational systems and that $(\Psi_-, \Psi_+)\colon R \to R'$ is a Tukey connection.*

(a) If $D' \subseteq Y'$ is R'-dominating, then $\Psi_+[D']$ is R-dominating.

(b) $(\Psi_+, \Psi_-)\colon (R')^\perp \to R^\perp$ is a Tukey connection.

(c) If $E \subseteq X$ is R-unbounded then $\Psi_-[E]$ is R'-unbounded.

Corollary 1.14. *(a) $R \preceq_{\mathrm{T}} R'$ implies $(R')^\perp \preceq_{\mathrm{T}} R^\perp$.*

(b) $R \preceq_{\mathrm{T}} R'$ implies $\mathfrak{b}(R') \leq \mathfrak{b}(R)$ and $\mathfrak{d}(R) \leq \mathfrak{d}(R')$.

(c) $R \cong_{\mathrm{T}} R'$ implies $\mathfrak{b}(R') = \mathfrak{b}(R)$ and $\mathfrak{d}(R) = \mathfrak{d}(R')$.

Example 1.15. The diagram in Figure 1 can be expressed in terms of the Tukey order since $\mathbf{C}_{\mathcal{I}} \preceq_{\mathrm{T}} \mathcal{I}$ and $\mathbf{C}_{\mathcal{I}}^\perp \preceq_{\mathrm{T}} \mathcal{I}$ when $\mathcal{I}$ is an ideal on X such that $\bigcup \mathcal{I} = X$. The first inequality is obtained via the Tukey connection $x \in X \mapsto \{x\} \in \mathcal{I}$ and $A \in \mathcal{I} \mapsto A \in \mathcal{I}$, and the second is obtained via $A \in \mathcal{I} \mapsto A \in \mathcal{I}$ and $B \in \mathcal{I} \mapsto y_B \in X$ such that $y_B \notin B$.

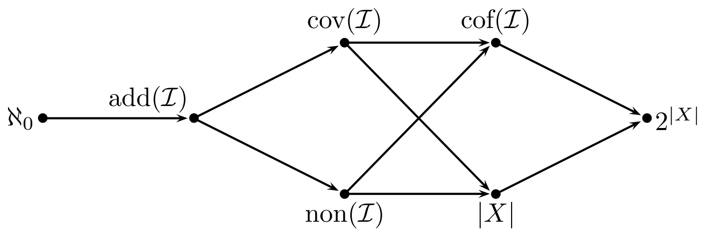

Figure 1: Diagram of the cardinal characteristics associated with $\mathcal{I}$. An arrow $\mathfrak{x} \to \mathfrak{y}$ means that (provably in ZFC) $\mathfrak{x} \leq \mathfrak{y}$.

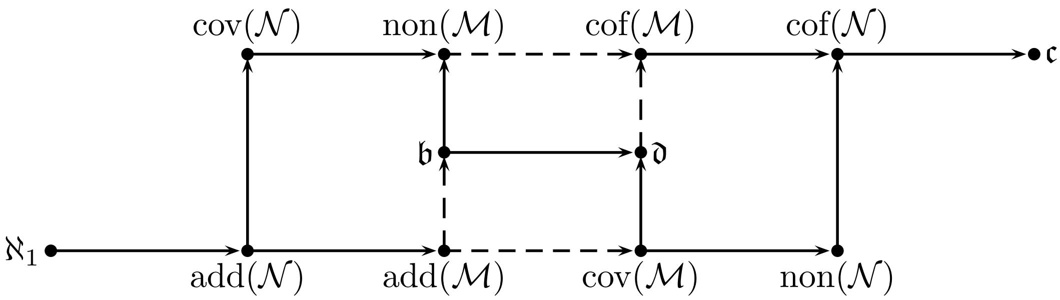

Figure 2: Cichoń's diagram. The arrows mean $\leq$ and dotted arrows represent $\mathrm{add}(\mathcal{M}) = \min\{\mathfrak{b}, \mathrm{cov}(\mathcal{M})\}$ and $\mathrm{cof}(\mathcal{M}) = \max\{\mathfrak{d}, \mathrm{non}(\mathcal{M})\}$, which we call the *dependent values*.

Cichoń's diagram (Figure 2) illustrates the inequalities between the cardinal characteristics associated with measure and category of the real numbers. The initial study of this diagram was completed between 1981 and 1993. Inequalities were proved by Bartoszyński, Fremlin, Miller, Rothberger and Truss. The name "Cichoń's diagram" was attributed by Fremlin [Fre83]. On the other hand, the diagram is complete in the sense that no more arrows can be added. Moreover, for any $\aleph_1$-$\aleph_2$ assignment to the cardinals in Cichoń's diagram that does not contradict the arrows (and the dependent values), there is a forcing poset that forces the corresponding model. This part of the study was completed by Bartoszyński, Judah, Miller and Shelah. In fact, the inequalities in Cichoń's diagram can be obtained via the Tukey connections as illustrated in Figure 3. See e.g. [BJ95, Bla10] for all the details.

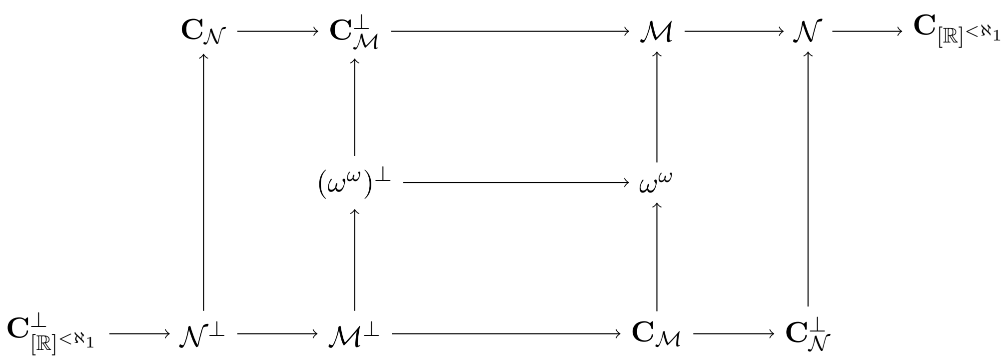

Figure 3: Cichon's diagram via Tukey connections. Any arrow represents a Tukey connection in the given direction.

We look at more classical cardinal characteristics. Concerning those associated with $\mathcal{E}$:

Lemma 1.16 ([BJ95, Lem. 7.4.3])**.** $\mathbf{C}_{\mathcal{E}} \preceq_{\mathrm{T}} \mathrm{Spl}$.

Theorem 1.17 ([BS92], see also [BJ95, Sec. 2.6])**.**

(a) $\min\{\mathfrak{b}, \mathrm{non}(\mathcal{N})\} \leq \mathrm{non}(\mathcal{E}) \leq \min\{\mathrm{non}(\mathcal{M}), \mathrm{non}(\mathcal{N})\}$.

(b) $\max\{\mathrm{cov}(\mathcal{M}), \mathrm{cov}(\mathcal{N})\} \leq \mathrm{cov}(\mathcal{E}) \leq \max\{\mathfrak{d}, \mathrm{cov}(\mathcal{N})\}$.

(c) $\mathrm{add}(\mathcal{E}) = \mathrm{add}(\mathcal{M})$ *and* $\mathrm{cof}(\mathcal{E}) = \mathrm{cof}(\mathcal{M})$.

Definition 1.18.

(1) For $a, b \in [\omega]^{\aleph_0}$, we define $a \subseteq^* b$ iff $a \smallsetminus b$ is finite;

(2) and we say that a *splits* b if both $a \cap b$ and $b \smallsetminus a$ are infinite, that is, $a \not\supseteq^* b$ and $\omega \smallsetminus a \not\supseteq^* b$.

(3) $F \subseteq [\omega]^{\aleph_0}$ is a *splitting family* if every $y \in [\omega]^{\aleph_0}$ is split by some $x \in F$. The *splitting number* $\mathfrak{s}$ is the smallest size of a splitting family.

(4) $D \subseteq [\omega]^{\aleph_0}$ is an *unreaping family* if no $x \in [\omega]^{\aleph_0}$ splits every member of D. The *reaping number* $\mathfrak{r}$ is the smallest size of an unreaping family.

(5) $D \subseteq [\omega]^{\aleph_0}$ is *groupwise dense* when:

 (i) if $a \in [\omega]^{\aleph_0}$, $b \in D$ and $a \subseteq^* b$, then $a \in D$,

(ii) if $\langle I_n : n < \omega \rangle$ is an interval partition of ω then $\bigcup_{n\in a} I_n \in D$ for some $a \in [\omega]^{\aleph_0}$.

The *groupwise density number* $\mathfrak{g}$ is the smallest size of a collection of groupwise dense sets whose intersection is empty.

(6) The *distributivity number* $\mathfrak{h}$ is the smallest size of a collection of dense subsets of $\langle [\omega]^{\aleph_0}, \subseteq^* \rangle$ whose intersection is empty.

(7) Say that $a \in [\omega]^{\aleph_0}$ is a *pseudo-intersection of* $F \subseteq [\omega]^{\aleph_0}$ if $a \subseteq^* b$ for all $b \in F$.

(8) The *pseudo-intersection number* $\mathfrak{p}$ is the smallest size of a filter base of subsets of $[\omega]^{\aleph_0}$ without pseudo-intersection.

(9) The *tower number* $\mathfrak{t}$ is the smallest length of a (transfinite) $\subseteq^*$-decreasing sequence in $[\omega]^{\aleph_0}$ without pseudo-intersection.

(10) Given a class $\mathcal{P}$ of forcing notions, $\mathfrak{m}(\mathcal{P})$ denotes the minimal cardinal κ such that, for some $Q \in \mathcal{P}$, there is some collection $\mathcal{D}$ of size κ of dense subsets of Q without a filter in Q intersecting every member of $\mathcal{D}$.

(11) Let $\mathbb{P}$ be a poset. A set $A \subseteq \mathbb{P}$ is *k-linked (in $\mathbb{P}$)* if every k-element subset of A has a lower bound in $\mathbb{P}$. A is *centered* if it is k-linked for all $k \in \omega$.

(12) A poset $\mathbb{P}$ is *k-Knaster*, if for each uncountable $A \subseteq \mathbb{P}$ there is a k-linked uncountable $B \subseteq A$. And $\mathbb{P}$ *has precaliber* $\aleph_1$, if such a B can be chosen centered. For notational convenience, 1-*Knaster* means ccc, and ω-*Knaster* means precaliber $\aleph_1$.

(13) For $1 \leq k \leq \omega$ denote $\mathfrak{m}_k := \mathfrak{m}(k\text{-Knaster})$ and $\mathfrak{m} := \mathfrak{m}_1$. We also set $\mathfrak{m}_0 := \aleph_1$.

(14) Define the relational system $\mathrm{Pred} = \langle \omega^\omega, \mathrm{Pr}, \sqsubset^{\mathrm{pr}} \rangle$ where Pr is the set of functions π (called *predictors*) into ω with domain $\bigcup_{n\in D_\pi} \omega^n$ for some $D_\pi \in [\omega]^{\aleph_0}$, and

$$x \sqsubset^{\mathrm{pr}} \pi \text{ iff } \exists m < \omega\ \forall n \geq m \colon n \in D_\pi \Rightarrow x(n) = \pi(x{\restriction}n),$$

in which case we say that π *predict* x. We define $\mathfrak{e} := \mathfrak{b}(\mathrm{Pred})$ the *evasion number*.

The inequalities between the cardinal characteristics presented so far are summarized in Figure 4. See [Bla10, BJ95] for the definitions and the proofs for the inequalities (with the exception of $\mathrm{cof}(\mathcal{M}) \leq \mathfrak{i}$, which was proved in [BHHH04]).

Below we list some additional properties of these cardinals. Unless noted otherwise, proofs can be found in [Bla10].

Fact 1.19.

(1) *In [MS16] it was proved that* $\mathfrak{p} = \mathfrak{t}$.[1]

(2) *The cardinals* $\mathrm{add}(\mathcal{N})$, $\mathrm{add}(\mathcal{M})$, $\mathfrak{b}$, $\mathfrak{t}$, $\mathfrak{h}$ *and* $\mathfrak{g}$ *are regular.*

(3) $\mathrm{cf}(\mathfrak{s}) \geq \mathfrak{t}$ *(see [DS18]).*

(4) $2^{<\mathfrak{t}} = \mathfrak{c}$.

(5) $\mathrm{cf}(\mathfrak{c}) \geq \mathfrak{g}$.

(6) *For* $1 \leq k \leq k' \leq \omega$, $\mathfrak{m}_k \leq \mathfrak{m}_{k'}$.

[1]Only the trivial inequality $\mathfrak{p} \leq \mathfrak{t}$ is used in this mini-course.

(7) *For* $1 \leq k \leq \omega$, $\mathfrak{m}_k > \aleph_1$ *implies* $\mathfrak{m}_k = \mathfrak{m}_\omega$ *(well-known, but see e.g. [GKMS21a, Lemma 4.2]).*

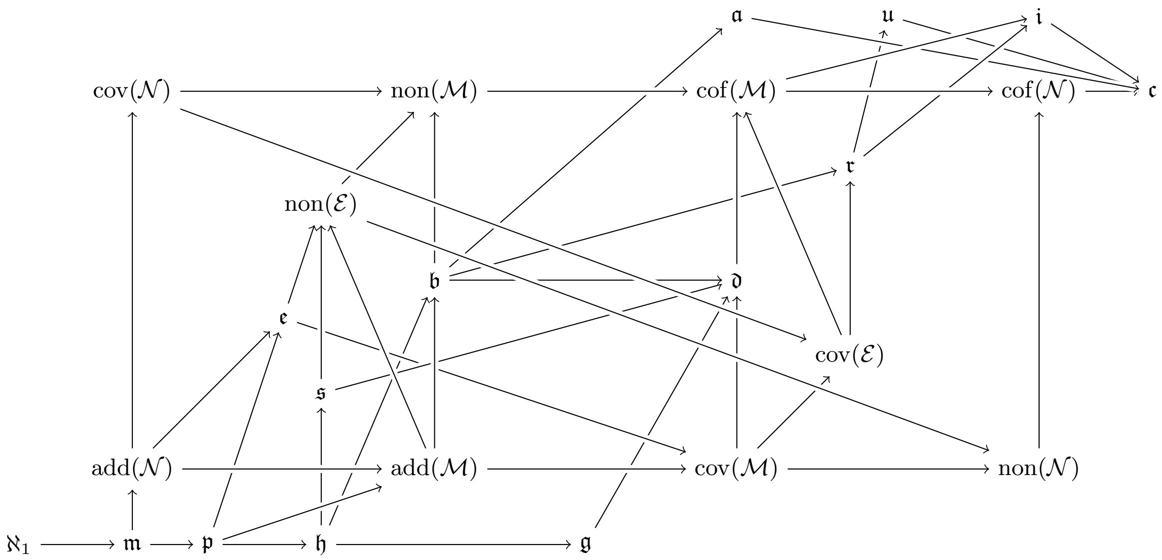

Figure 4: Diagram of inequalities between classical cardinal characteristics

Concerning cofinalities:

Fact 1.20. *Let $\mathcal{I}$ be an ideal on X such that $\bigcup \mathcal{I} = X$.*

(1) $\mathrm{add}(\mathcal{I})$ *is regular,* $\mathrm{cf}(\mathrm{cof}(\mathcal{I})) \geq \mathrm{add}(\mathcal{I})$ *and* $\mathrm{cf}(\mathrm{non}(\mathcal{I})) \geq \mathrm{add}(\mathcal{I})$*.*

(2) $\mathrm{cf}(\mathrm{cov}(\mathcal{M})) \geq \mathrm{add}(\mathcal{N})$ *(Bartoszyński and Shelah 1989, see [BJ95, Thm. 5.1.5]).*

(3) *If* $\mathrm{cov}(\mathcal{N}) \leq \mathfrak{b}$ *then* $\mathrm{cf}(\mathrm{cov}(\mathcal{N})) > \omega$ *(Bartoszyński 1988, see [BJ95, Thm. 5.1.17]).*

(4) *If* $\mathrm{cov}(\mathcal{E}) \leq \mathfrak{d}$ *then* $\mathrm{cf}(\mathrm{cov}(\mathcal{E})) > \omega$ *(Miller, see [BJ95, Thm. 5.1.18]).*

The problem of the cofinality of $\mathrm{cov}(\mathcal{N})$ was settled with the following result.

Theorem 1.21 (Shelah [She00])**.** *It is consistent with* ZFC *that* $\mathrm{cf}(\mathrm{cov}(\mathcal{N})) = \omega$*.*

The following question is still unsolved.

Question 1.22. *Is it consistent with* ZFC *that* $\mathrm{cf}(\mathrm{cov}(\mathcal{E})) = \omega$*?*

To solve this problem in the positive, it is necessary to force $\mathfrak{d} < \mathrm{cov}(\mathcal{E})$, which implies $\mathrm{cov}(\mathcal{E}) = \mathrm{cov}(\mathcal{N})$ (see Theorem 1.17), so it would be needed to force $\mathrm{cf}(\mathrm{cov}(\mathcal{N})) = \omega$ via an ω^ω-bounding forcing.

Figure 4 is quite complete, but the following is still unknown.

Question 1.23. *Is* $\mathfrak{a} \leq \mathfrak{i}$*?*

It is not even known how to solve:

Question 1.24. *Does* $\mathfrak{i} = \aleph_1$ *imply* $\mathfrak{a} = \aleph_1$*?*

The positive answer to this problem is implied by the positive answer to the following famous problem in set theory.

Question 1.25 (Roitman's problem)**.** *Does* $\mathfrak{d} = \aleph_1$ *imply* $\mathfrak{a} = \aleph_1$*?*

The following strengthening of Roitman's problem was formulated by Brendle and Raghavan [BR14].

Question 1.26. *Does* $\mathfrak{b} = \mathfrak{s} = \aleph_1$ *imply* $\mathfrak{a} = \aleph_1$*?*

2 Finite Support iterations

2.1 Generic reals

We first look at the types of generic reals we intend to add by forcing. Recall that a *Polish space* is a separable completely metrizable space. The real line $\mathbb{R}$ and any product $\prod_{n<\omega} b(n)$ of countable discrete spaces, such as the Cantor space 2^ω and the Baire space ω^ω, are canonical examples. Polish spaces are used to resemble the combinatorics and the descriptive set theory of the real line.

For a Polish space Z, denote by $\bar{\mathbf{\Sigma}}(Z)$ the field of sets generated by the analytic subsets of Z.

Definition 2.1. We say that $R = \langle X, Y, \sqsubset\rangle$ is a *relational system of the reals* if

(i) $X \in \bar{\mathbf{\Sigma}}(Z_1)$ and $Y \in \bar{\mathbf{\Sigma}}(Z_2)$ for some Polish spaces Z_1 and Z_2, and

(ii) $\sqsubset \in \bar{\mathbf{\Sigma}}(Z_1 \times Z_2)$.

In most of the cases, $X = Z_1$ is a perfect Polish space and, for any $y \in Y$, $\{x \in X : x \sqsubset y\}$ is meager in X.

The reason we use $\bar{\mathbf{\Sigma}}$ is to have absoluteness of the statements "$x \in X$", "$y \in Y$" and "$x \sqsubset y$". In general, we can just use definable sets X, Y and $\sqsubset$ such that the previous statements are absolute for the arguments we are carrying out.

For the rest of this section, we fix a relational system of the reals $R = \langle X, Y, \sqsubset\rangle$. We introduce the following type of (generic) reals related to R.

Definition 2.2. Let M be a (transitive) model of ZFC.[2]

(1) A point $y^* \in Y$ is *R-dominating over M* if $\forall x \in X \cap M \colon x \sqsubset y^*$.

(2) A point $x^* \in X$ is *R-unbounded over M* if $\forall y \in Y \cap M \colon x^* \not\sqsubset y$.

[2]Since such set models cannot exist, most of the time this expression means that M satisfies a large enough fragment of ZFC to perform the arguments at hand.

We look at many particular cases related to the cardinals in Cichoń's diagram.

Definition 2.3. [Localization] For $h \in \omega^\omega$ and $H \subseteq \omega^\omega$, define

$$\mathcal{S}(\omega, h) := \prod_{i<\omega} [\omega]^{\leq h(i)},$$
$$\mathcal{S}(\omega, H) := \bigcup_{h \in H} \mathcal{S}(\omega, h).$$

Objects in these sets are usually called *slaloms*.

For functions x and y with domain ω, we define the relation "y *localizes* x" by

$$x \in^* y \text{ iff } \exists\, m < \omega\ \forall\, i \geq m \colon x(i) \in y(i).$$

Define the following *localization relational systems*:

$$\mathbf{Lc}(\omega, h) := \langle \omega^\omega, \mathcal{S}(\omega, h), \in^* \rangle,$$
$$\mathbf{Lc}(\omega, H) := \langle \omega^\omega, \mathcal{S}(\omega, H), \in^* \rangle.$$

It is easy to check that these are relational systems of the reals when H is countable.

The localization relational systems work to easily characterize the cardinal characteristics associated with $\mathcal{N}$.

Theorem 2.4 (Bartoszyński [Bar84](1984), see also [CM23, Sec. 4])**.** *If $h \to \infty$ and $H \subseteq \omega^\omega$ is a countable set containing some function diverging to infinity, then*

$$\mathbf{Lc}(\omega, h) \cong_{\mathrm{T}} \mathbf{Lc}(\omega, H) \cong_{\mathrm{T}} \mathcal{N}.$$

In particular, $\mathfrak{b}(\mathbf{Lc}(\omega, h)) = \mathfrak{b}(\mathbf{Lc}(\omega, H)) = \mathrm{add}(\mathcal{N})$ *and* $\mathfrak{d}(\mathbf{Lc}(\omega, h)) = \mathfrak{d}(\mathbf{Lc}(\omega, H)) = \mathrm{cof}(\mathcal{N})$.

We now introduce a forcing to modify $\mathbf{Lc}(\omega, h)$. In the context of forcing, V always refers to the *ground model*.

Definition 2.5 (Localization forcing)**.** For $h \in \omega^\omega$, define the poset[3]

$$\mathbb{L}\mathrm{c}_h := \{(n, \varphi) \in \omega \times \mathcal{S}(\omega, h) \colon \exists\, m < \omega \colon \varphi \in \mathcal{S}(\omega, m)\}$$

ordered by

$$(n', \varphi') \leq (n, \varphi) \text{ iff } n \leq n',\ \ \varphi' {\restriction} n = \varphi {\restriction} n \text{ and } \forall\, i < \omega \colon \varphi(i) \subseteq \varphi'(i).$$

When $h \to \infty$ we have that $\mathbb{L}\mathrm{c}_h$ is ccc (even σ-k-linked for any $k < \omega$) and it adds a generic slalom $\varphi_* \in \mathcal{S}(\omega, h)$ which localizes all functions in the ground model, i.e. it is $\mathbf{Lc}(\omega, h)$-dominating over the ground model. If G is $\mathbf{Lc}(\omega, h)$-generic over V, the generic slalom is defined by $\varphi_*(i) := \varphi(i)$ when $(n, \varphi) \in G$ and $i < n$ (this value is the same for any such (n, φ)).

[3]The m in $\mathcal{S}(\omega, m)$ refers to the constant function with value m.

We present a relational system of the reals that represents the relational system $\mathbf{C}_\mathcal{N}$ (more precisely, its dual). For this purpose, we code measure zero sets as follows.

Definition 2.6. For any topological space X, denote by $\mathcal{B}(X)$ the σ-algebra of Borel subsets of X. Let Lb_2 be the measure on $\mathcal{B}(2^\omega)$ defined as the product measure of the uniform measure on $2 = \{0,1\}$.[4] Recall that $\{[s] \colon s \in 2^{<\omega}\}$ forms a base of 2^ω and that each $[s]$ is clopen in 2^ω. Then, Lb_2 is the unique measure on $\mathcal{B}(2^\omega)$ such that $\mathrm{Lb}_2([s]) = 2^{-|s|}$ for any $s \in 2^{<\omega}$.[5]

We abuse notation and denote $[F] := \bigcup_{s\in F}[s]$ for $F \subseteq 2^{<\omega}$. Since 2^ω is compact, we have that the clopen sets are precisely of the form $[c]$ for $c \subseteq 2^{<\omega}$ finite.

We code measure zero subsets of 2^ω in the following way. Fix a sequence $\bar\varepsilon = \langle \varepsilon_n \colon n < \omega\rangle$ of positive real numbers such that $\sum_{n<\omega}\varepsilon_n < \infty$. Define

$$\Omega_{\bar\varepsilon} := \left\{\bar c = \langle c_n \colon n<\omega\rangle \colon \forall\, n<\omega \colon c_n \in [2^{<\omega}]^{<\aleph_0} \text{ and } \mathrm{Lb}_2([c_n]) < \varepsilon_n\right\}.$$

For any sequence $\bar c = \langle c_n \colon n < \omega\rangle$ of finite subsets of $2^{<\omega}$, denote

$$N(\bar c) = \bigcap_{m<\omega}\bigcup_{n\geq m}[c_n],$$

i.e. for $x \in 2^\omega$, $x \in N(\bar c)$ iff $x \in [c_n]$ for infinitely many n.

Define the relational system $\mathrm{Cn} := \langle \Omega_{\bar\varepsilon}, 2^\omega, \not\blacktriangleright\rangle$ such that $\bar c \not\blacktriangleright y$ iff $y \notin N(\bar c)$.

The sequences in $\Omega_{\bar\varepsilon}$ are simple codes of (a base of) measure zero sets in 2^ω.

Fact 2.7 (See e.g. [BJ95, Lemma 2.3.10])**.** *If $\bar c \in \Omega_{\bar\varepsilon}$, then $N(\bar c) \in \mathcal{N}(\mathrm{Lb}_2)$ and, for any $A \in \mathcal{N}(\mathrm{Lb}_2)$, there exists $\bar c \in \Omega_{\bar\varepsilon}$ such that $A \subseteq N(\bar c)$.*

In combinatorics of the reals, it is the same (and more practical) to work in the Cantor space than on $\mathbb{R}$, because functions in 2^ω represent the numbers in $[0,1]$ when expressed in base 2. For this reason, the measure theory of 2^ω is equivalent to the one of $[0,1]$ (with the Lebesgue measure), so $\mathcal{N}(\mathbb{R}) \cong_{\mathrm{T}} \mathcal{N}(2^\omega)$ and $\mathbf{C}_{\mathcal{N}(\mathbb{R})} \cong_{\mathrm{T}} \mathbf{C}_{\mathcal{N}(2^\omega)}$. See details in [Lev02, Ch. VII, §3].

As a direct consequence of Fact 2.7, we obtain:

Fact 2.8. $\mathrm{Cn} \cong_{\mathrm{T}} \mathbf{C}^\perp_{\mathcal{N}(2^\omega)}$, *so* $\mathfrak{b}(\mathrm{Cn}) = \mathrm{cov}(\mathcal{N})$ *and* $\mathfrak{d}(\mathrm{Cn}) = \mathrm{non}(\mathcal{N})$.

Definition 2.9. Random forcing is $\mathcal{B}(2^\omega) \smallsetminus \mathcal{N}(2^\omega)$ ordered by $\subseteq$. If G is a generic set over V, then we can define $r \in 2^\omega$ by $r := \bigcup\{s \in 2^{<\omega} \colon [s] \in G\}$. Such r is called a *random real (over V)*.

Random forcing is ccc (even σ-k-linked for any $k < \omega$).

Fact 2.10. *If r is a random real over V, then $r \notin N(\bar c)$ for any $\bar c \in \Omega_{\bar\varepsilon} \cap V$, i.e. any random real over V is* Cn*-dominating over V.*

[4] The uniform measure on a finite non-empty set set b assigns probability $\frac{1}{|b|}$ to each point.

[5] For $s \in 2^{<\omega}$, $|s| = |\operatorname{dom} s|$ is the length of s as a sequence.

The previous indicates that any random real over V evades the Borel measure zero sets coded in the ground model.

Concerning the directed preorder $\langle\omega^\omega, \leq^*\rangle$, we introduce:

Definition 2.11. *Hechler forcing* is the poset $\mathbb{D} := \omega^{<\omega} \times \omega^\omega$ ordered by

$$(t, g) \leq (s, f) \text{ iff } s \subseteq t,\ \forall\, i < \omega\colon f(i) \leq g(i), \text{ and } \forall\, i \in |t| \smallsetminus |s|\colon t(i) \geq f(i).$$

This poset is ccc (even σ-centered).

If G is $\mathbb{D}$-generic over V, then $d := \bigcup\{s\colon \exists\, f\colon (s, f) \in G\}$ is $\langle\omega^\omega, \leq^*\rangle$-dominating over V.

We now turn to $\mathbf{C}_{\mathcal{M}}$. First, we introduce a useful characterization of its cardinal characteristics.

Definition 2.12. Define the relation system $\mathrm{Ed} := \langle\omega^\omega, \omega^\omega, \neq^*\rangle$ where

$$x \neq^* y \text{ iff } \exists\, m < \omega\ \forall\, i \geq m\colon x(i) \neq y(i).$$

Theorem 2.13 (Miller [Mil82], Bartoszyński [Bar87], see also [CM23, Thm. 5.3])**.** $\mathfrak{b}(\mathrm{Ed}) = \mathrm{non}(\mathcal{M})$ *and* $\mathfrak{d}(\mathrm{Ed}) = \mathrm{cov}(\mathcal{M})$.

Definition 2.14. Define the *eventually different real forcing* by

$$\mathbb{E} := \left\{(s, \varphi)\colon s \in \omega^{<\omega},\ \varphi \in \bigcup_{m<\omega} \mathcal{S}(\omega, m)\right\}$$

ordered by

$$(t, \psi) \leq (s, \varphi) \text{ iff } s \subseteq t,\ \forall\, i < \omega\colon \varphi(i) \leq \psi(i), \text{ and } \forall\, i \in |t| \smallsetminus |s|\colon t(i) \notin \varphi(i).$$

This forcing is ccc (even σ-centered).

If G is $\mathbb{E}$-generic over V then $e := \bigcup\{s\colon \exists\, \varphi\colon (s, \varphi) \in G\}$ is Ed-dominating over V.

Therefore, by Theorem 2.13, $\mathbb{E}$ can be used to increase $\mathrm{non}(\mathcal{M})$. But it actually does more:

Theorem 2.15 (Cardona & Mejía)**.** *$\mathbb{E}$ adds a $\mathbf{C}_{\mathcal{E}}$-dominating real over V, i.e. an F_σ subset of $\mathbb{R}$ that covers $\mathbb{R} \cap V$.*

In the same way as measure, we have that $\mathcal{M}(\mathbb{R}) \cong_{\mathrm{T}} \mathcal{M}(2^\omega)$ and $\mathbf{C}_{\mathcal{M}(\mathbb{R})} \cong_{\mathrm{T}} \mathbf{C}_{\mathcal{M}(2^\omega)}$, so we obtain the same cardinal characteristics for the meager ideal using the Cantor space instead of $\mathbb{R}$. More generally, as a consequence of [Kec95, Subsec. 15.D]:

Theorem 2.16. *For any perfect Polish space X, $\mathcal{M}(X) \cong_{\mathrm{T}} \mathcal{M}(\mathbb{R})$ and $\mathbf{C}_{\mathcal{M}(X)} \cong_{\mathrm{T}} \mathbf{C}_{\mathcal{M}(\mathbb{R})}$.*

We now look at the effect of Cohen forcing to meager sets. As we did with measure zero, we introduce a coding of (a base of) meager subsets of 2^ω.

Definition 2.17. Let $\mathbb{I}$ be the set of interval partitions of ω. Define the relational system $\mathrm{M} := \langle 2^\omega, 2^\omega \times \mathbb{I}, \sqsubset^{\mathrm{m}} \rangle$ where

$$x \sqsubset^{\mathrm{m}} (y, \bar{I}) \text{ iff } \exists\, m < \omega\ \forall\, n \geq m \colon x{\restriction}I_n \neq y{\restriction}I_n.$$

The members of $2^\omega \times \mathbb{I}$ are usually called *matching reals*. For any matching real $(y, \bar{I})$, define $M(y, \bar{I}) := \{x \in 2^\omega \colon x \sqsubset^{\mathrm{m}} (y, \bar{I})\}$.

Fact 2.18 (See e.g. [Bla10]). *For any matching real $(y, \bar{I})$, $M(y, \bar{I})$ is meager in 2^ω. And, for any $A \in \mathcal{M}(2^\omega)$, there is some matching real $(y, \bar{I})$ such that $A \subseteq M(y, \bar{I})$.*

Corollary 2.19. $\mathrm{M} \cong_{\mathrm{T}} \mathbf{C}_{\mathcal{M}}(2^\omega)$. *In particular,* $\mathfrak{b}(\mathrm{M}) = \mathrm{non}(\mathcal{M})$ *and* $\mathfrak{d}(\mathrm{M}) = \mathrm{cov}(\mathcal{M})$.

Definition 2.20. Let I be a set and $\bar{b} = \langle b(i) \colon i \in I \rangle$ a sequence of non-empty sets. Define the poset

$$\mathrm{Fn}(\bar{b}) := \{p \colon p \text{ is a finite function, } \operatorname{dom} p \subseteq I \text{ and } \forall\, i \in \operatorname{dom} p \colon p(i) \in b(i)\}$$

ordered by $\supseteq$. The generic real added by this poset is $g := \bigcup G \in \prod \bar{b} := \prod_{i \in I} b(i)$ whenever G is $\mathrm{Fn}(I, \bar{b})$-generic over V.

We use this forcing to add Cohen reals, not just over 2^ω or ω^ω, but over any perfect space of the form $\prod_{n<\omega} b(n)$, endowed with the product topology for countable discrete spaces $b(n)$ $(n < \omega)$.

Fix a countable sequence $\bar{b} := \langle b(n) \colon n < \omega \rangle$ of countable non-empty sets. Note that $\prod \bar{b}$ is a perfect Polish space iff $|b(n)| \geq 2$ for infinitely many $n < \omega$. In this case, we call $\mathrm{Fn}(\bar{b})$ the *forcing adding a Cohen real in* $\prod \bar{b}$, usually referred to as *Cohen forcing*. We use c to denote the generic real in $\prod \bar{b}$ added by this poset, which we often call *Cohen real*. For example, $\Omega_{\bar{\varepsilon}}$ is such a space, and a Cohen real in $\Omega_{\bar{\varepsilon}}$ over V codes a measure zero set that covers $2^\omega \cap V$. The letter $\mathbb{C}$ is reserved for any version of Cohen forcing.

For any set I, denote $\mathbb{C}_I := \mathrm{Fn}(\hat{b})$ where $\hat{b} := \langle b(i,n) \colon i \in I,\ n < \omega \rangle$ is defined by $b(i,n) := b(n)$. This poset adds a sequence $\langle c_i \colon i \in I \rangle$ where each $c_i \in \prod_{n<\omega} b(n)$ is a Cohen real over V (and even over $V^{\mathbb{C}_{I \smallsetminus \{i\}}}$).

All the versions of Cohen forcing are forcing equivalent:

Theorem 2.21. *Any countable atomless forcing notion is forcing equivalent with* $\mathbb{C}$.

In general, for any perfect Polish space X, it is possible to define a countable atomless forcing that adds a generic real $c \in X$.[6] The main property of this generic real is that it evades all the Borel meager subsets of X coded in the ground model. In particular,

Theorem 2.22. *If $c \in 2^\omega$ is a Cohen real over V, then $c \notin M(y, \bar{I})$ for any matching real $(y, \bar{I}) \in V$. In particular, any Cohen real is* M*-unbounded over V.*

[6] Using finite fragments of Cauchy sequences coming from a countable dense subset of X.

2.2 FS iterations

We now turn to FS (finite support) iterations. Any FS iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \colon \alpha \leq \pi,\ \beta < \pi \rangle$ of length π is defined by recursion as follows:

(I) $\mathbb{P}_0 := \{\langle\ \rangle\}$ is the poset containing the empty sequence $\langle\ \rangle$, usually called the *trivial poset*.

(II) When $\mathbb{P}_\alpha$ has been defined, we pick a $\mathbb{P}_\alpha$-name $\dot{\mathbb{Q}}_\alpha$ of a poset and define $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}_\alpha$.

(III) For limit $\gamma \leq \pi$, $\mathbb{P}_\gamma := \operatorname{limdir}_{\alpha<\gamma} \mathbb{P}_\alpha = \bigcup_{\alpha<\gamma} \mathbb{P}_\alpha$ ordered by

$$q \leq_\gamma p \text{ iff } \exists\, \alpha < \gamma \colon p, q \in \mathbb{P}_\alpha \text{ and } q \leq_\alpha p.$$

Here, $\leq_\alpha$ denotes the preorder of $\mathbb{P}_\alpha$. It can be proved by induction that $\mathbb{P}_\alpha \subsetdot \mathbb{P}_\beta$ whenever $\alpha \leq \beta \leq \pi$, where $\subsetdot$ denotes the *complete-subposet* relation.[7]

If G is $\mathbb{P}_\pi$-generic over V and $\alpha \leq \pi$, then $G_\alpha := \mathbb{P}_\alpha \cap G$ is $\mathbb{P}_\alpha$-generic over V, so $G_\pi = G$. In the context of FS iterations, we denote $V_\alpha := V[G_\alpha]$, so $V_0 = V$. The relation $\subsetdot$ indicates that $V_\alpha \subseteq V_\beta$ whenever $\alpha \leq \beta \leq \pi$. So, when $\alpha < \pi$, we call V_α an *intermediate generic extension*, and V_π the *final generic extension*.

In this context, we abbreviate the forcing relation $\Vdash_{\mathbb{P}_\alpha}$ by $\Vdash_\alpha$.

We review some basic facts about FS iterations of ccc posets.

Lemma 2.23. *Any FS iteration of ccc posets is ccc, i.e. if $\Vdash_\beta \dot{\mathbb{Q}}_\beta$ is ccc for all $\beta < \pi$, then $\mathbb{P}_\alpha$ is ccc for all $\alpha \leq \pi$.*

Lemma 2.24. *In any FS iteration of ccc posets of length π: if $\operatorname{cf}(\pi) > \omega$ then $\mathbb{R} \cap V_\pi = \bigcup_{\alpha<\pi} \mathbb{R} \cap V_\alpha$.*

Lemma 2.25. *Any FS iteration of non-trivial*[8] *posets adds Cohen reals at limit stages. Concretely, $\mathbb{P}_{\alpha+\omega}$ adds a Cohen real over V_α.*

The Cohen reals added by a FS iteration determine a Tukey connection for $\mathbf{C}_\mathcal{M}$ as follows.

Corollary 2.26. *Any FS iteration of ccc posets of length π with uncountable cofinality forces $\operatorname{non}(\mathcal{M}) \leq \operatorname{cf}(\pi) \leq \operatorname{cov}(\mathcal{M})$, even more, $\pi \preceq_{\mathrm{T}} \mathbf{C}_\mathcal{M}$.*

Proof. Work in V_π. For any matching real $(y, \bar{I})$, by Lemma 2.24 there is some $\alpha_{y,\bar{I}} < \pi$ such that $(y, \bar{I}) \in V_{\alpha_{y,\bar{I}}}$. On the other hand, by Lemma 2.25, there is some Cohen real $c_\alpha \in 2^\omega \cap V_{\alpha+\omega}$ over V_α. Then by Theorem 2.22, $c_\alpha \not\sqsubset^{\mathrm{m}} (y, \bar{I})$ whenever $(y, \bar{I}) \in V_\alpha$, which happens when $\alpha_{y,\bar{I}} \leq \alpha$. This indicates that the maps $\alpha \mapsto c_\alpha$ and $(y, \bar{I}) \mapsto \alpha_{y,\bar{I}}$ form a Tukey connection for $\pi \preceq_{\mathrm{T}} \mathrm{M}$. □

[7] $\mathbb{P} \subsetdot \mathbb{Q}$ iff $\mathbb{P}$ is a suborder of $\mathbb{Q}$, the incompatibility relation is preserved, and any predense subset of $\mathbb{P}$ is predense in $\mathbb{Q}$.

[8] A poset is *trivial* if all its conditions are pairwise compatible. This is equivalent to saying that the poset if forcing equivalent with the trivial poset.

The previous result puts a restriction to the models of Cichoń's diagram that can be obtained via FS iterations of ccc posets (of uncountable cofinality), since they force the inequality $\mathrm{non}(\mathcal{M}) \leq \mathrm{cov}(\mathcal{M})$. Therefore, in such models, the diagram of cardinal characteristics presented in Figure 4 takes the form as in Figure 5.

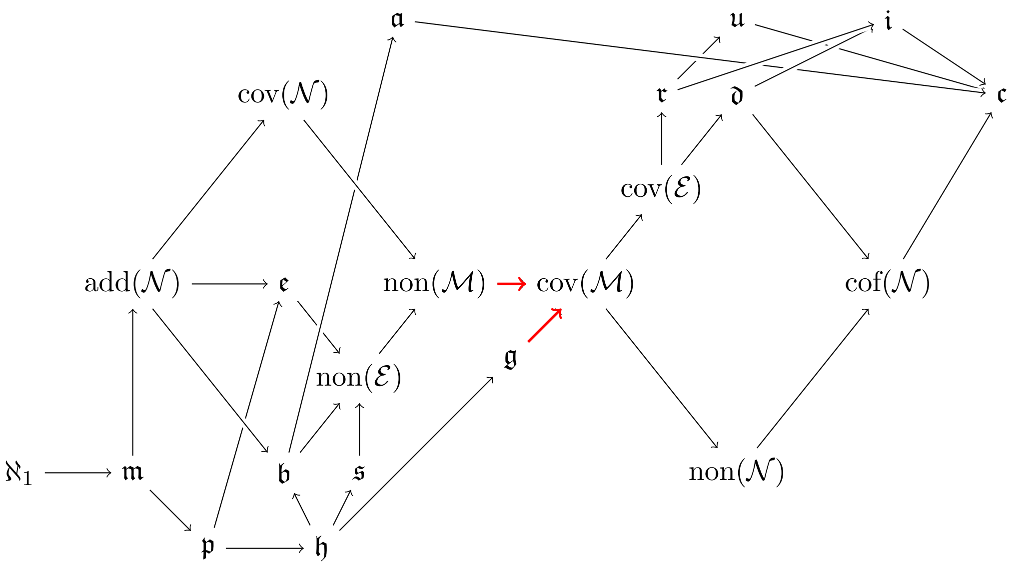

Figure 5: Cichoń's diagram with other classical cardinal characteristics after a FS iteration of ccc (non-trivial) posets of length with uncountable cofinality, as an effect of the forced inequalities $\mathrm{non}(\mathcal{M}) \leq \mathrm{cov}(\mathcal{M})$ and $\mathfrak{g} \leq \mathrm{cov}(\mathcal{M})$.

Below, we summarize the effect of the forcings introduced in this section to modify the cardinals in Cichoń's diagram:

(1) When $h \to \infty$, $\mathbb{L}\mathrm{c}_h$ adds $\mathrm{Lc}(\omega, h)$-dominating reals (so it affects $\mathrm{add}(\mathcal{N})$ and $\mathrm{cof}(\mathcal{N})$).

(2) Random forcing adds Cn-dominating reals (affecting $\mathrm{cov}(\mathcal{N})$ and $\mathrm{non}(\mathcal{N})$).

(3) Hechler forcing adds $\langle \omega^\omega, \leq^* \rangle$-dominating reals (affecting $\mathfrak{b}$ and $\mathfrak{d}$).

(4) The forcing $\mathbb{E}$ adds Ed-dominating reals, and also $\mathbf{C}_\mathcal{E}$-dominating reals (affecting $\mathrm{non}(\mathcal{M})$, $\mathrm{non}(\mathcal{E})$ and $\mathrm{cov}(\mathcal{M})$, $\mathrm{cov}(\mathcal{E})$).

(5) Cohen forcing adds M-unbounded reals (affecting $\mathrm{cov}(\mathcal{M})$ and $\mathrm{non}(\mathcal{M})$).

We are going to use these forcings to modify the cardinals in Cichońs diagram. However, we cannot just simply add dominating reals without any particular restriction, as indicated in the following result.

Lemma 2.27. *Let $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \colon \alpha \leq \pi,\ \beta < \pi \rangle$ be a FS iteration of ccc posets. Assume that $\mathrm{cf}(\pi) > \omega$, $K \subseteq \pi$ is cofinal, $R = \langle X, Y, \sqsubset \rangle$ is a relational system of the reals (see Definition 2.1), and assume that, for $\alpha \in K$, $\dot{\mathbb{Q}}_\alpha$ adds an R-dominating real over V_α.*

Then $\mathbb{P}_\pi$ forces $\mathfrak{d}(R) \leq \mathrm{cf}(\pi) \leq \mathfrak{b}(R)$, even $R \preceq_{\mathrm{T}} \pi$.

Moreover, if ZFC proves $\mathbf{C}_\mathcal{M} \preceq_{\mathrm{T}} R$, then $\mathbb{P}_\pi$ forces $R \cong_{\mathrm{T}} \mathbf{C}_\mathcal{M} \cong_{\mathrm{T}} \pi$, so $\mathfrak{b}(R) = \mathfrak{d}(R) = \mathrm{non}(\mathcal{M}) = \mathrm{cov}(\mathcal{M}) = \mathrm{cf}(\pi)$.

Proof. Work in V_π. If $x \in X$, by Lemma 2.24 there is some $\alpha_x < \pi$ such that $x \in V_{\alpha_x}$. On the other hand, for any $\alpha < \pi$, there is some $\beta_\alpha \in K$ above α, so $\dot{\mathbb{Q}}_{\beta_\alpha}$ adds an R-dominating real $y_\alpha \in Y$ over V_{β_α}. Then the maps $x \mapsto \alpha_x$ and $\alpha \mapsto y_\alpha$ form the Tukey connection for $R \preceq_{\mathrm{T}} \pi$.

There rest is consequence of Corollary 2.26. □

When aiming to force many different values to cardinal characteristics, we cannot add *full* dominating reals as in the previous lemma. However, there is a way to add *restricted* dominating reals, allowing better control of the cardinal characteristics. We develop this technique in the following part.

2.3 Book-keeping arguments

Fix, for the rest of this section:

(1) A relational system $R = \langle X, Y, \sqsubset \rangle$ of the reals (see Definition 2.1) such that $|X| = \mathfrak{c} = 2^{\aleph_0}$

(2) A very definable (i.e. Suslin) ccc poset $\mathbb{Q}_R$ adding R-dominating reals over the ground model, such that $|\mathbb{Q}_R| \leq \mathfrak{c}$. Note that $\mathbb{L}\mathrm{c}_h$, $\mathbb{D}$, $\mathbb{E}$, random forcing and Cohen forcing satisfy these conditions (for certain R as in the previous subsection).

(3) An infinite cardinal θ.

We aim to force $\mathfrak{b}(R) = \theta$. For the rest of this section we deal with $\mathfrak{b}(R) \geq \theta$, and from the following section we deal with the converse inequality.

Forcing $\mathfrak{b}(R) \geq \theta$ means to force that $\forall F \in [X]^{<\theta}\ \exists y \in Y\ \forall x \in F\colon x \sqsubset y$. One way is to deal with one F at a time along a FS iteration. Concretely, if we are at step α of a FS iteration, we pick some $F_\alpha \in [X]^{<\theta} \cap V_\alpha$ and aim to add a $y_\alpha \in Y \cap V_{\alpha+1}$ that R-dominates all members of F_α. A very effective idea to do this comes from Brendle: in V_α, pick a transitive model N_α of ZFC such that $F_\alpha \subseteq N_\alpha$ and $|N_\alpha| = \max\{\aleph_0, |F_\alpha|\} < \theta$.[9] So forcing with $\mathbb{Q}_\alpha = \mathbb{Q}_R^{N_\alpha}$ (which is ccc) does the job: it adds an R-dominating y_α over N_α, hence it dominates all members of F_α. The hope is that this y_α does not dominates much larger fragments of X.

Now, assume that π is an ordinal of uncountable cofinality, and that we perform a FS iteration of ccc posets of length π as explained before. To force $\mathfrak{b}(R) \geq \theta$, it is enough to guarantee that, in V_π, $\{F_\alpha\colon \alpha < \pi\}$ is *cofinal in* $[X]^{<\theta}$. Indeed, if $F \in [X]^{<\theta}$ then $F \subseteq F_\alpha$ for some $\alpha < \pi$, so y_α dominates all members of F_α, and then all members of F.

In the practice, we do not use all steps $\alpha < \pi$ to take care of $\mathfrak{b}(R) \geq \theta$, but only at steps $\alpha \in K$ for some (cofinal) $K \subseteq \pi$, while in other steps can be used to take care of something else. So we explain how to construct an iteration as above ensuring that some choice of $\{F_\alpha\colon \alpha \in K\}$ is cofinal in $[X]^{<\theta}$.

To do this, we first have to look at what happens to $|X|$ in the final extension. Recall that $|X| = \mathfrak{c}$. Assume $\theta \leq \lambda = \lambda^{\aleph_0}$. Then, in a FS iteration of length λ, we can ensure

[9] This is possible because the members of X are "reals".

that $|\mathbb{P}_\alpha| \leq \lambda$ and $\Vdash_\alpha \mathfrak{c} \leq \lambda$ as long as we have $\Vdash_\alpha |\dot{\mathbb{Q}}_\alpha| \leq \lambda$ for all $\alpha < \lambda$. This is fine in the context of this text because all forcings we use to iterate have size $\leq\mathfrak{c}$.

Therefore, in V_λ, $|X| = \mathfrak{c} = \lambda$, so $\mathrm{cof}([X]^{<\theta}) = \mathrm{cof}([\lambda]^{<\theta})$. Now, producing a collection $\{F_\alpha\colon \alpha \in K\}$ cofinal in $[X]^{<\theta}$ for some $K \subseteq \lambda$, $K \in V$, implies that $\mathrm{cof}([\lambda]^{<\theta}) = \mathrm{cof}([X]^{<\theta}) \leq |K| \leq \lambda$. Hence, a requirement to obtain such a cofinal family is that $\mathrm{cof}([\lambda]^{<\theta}) = \lambda$ and $|K| = \lambda$.

We now show that the assumptions $\theta \leq \mathrm{cf}(\lambda) \leq \lambda = \lambda^{\aleph_0}$ and $\mathrm{cof}([\lambda]^{<\theta}) = \lambda$ are enough to construct such an iteration via a *book-keeping argument*. Let $K \subseteq \lambda$ of size λ and fix a bijection $h\colon K \to \lambda \times \lambda$ such that $h(\alpha) = (\xi, \eta)$ implies $\xi \leq \alpha$. Now, perform a FS iteration of ccc posets and assume we have reached the stage $\alpha < \lambda$. Since $\mathrm{cov}([\lambda]^{<\theta})$ is not modified by ccc forcing[10], in V_α we can pick a cofinal $\{F_{\alpha,\eta}\colon \eta < \lambda\}$ on $[X]^{<\theta} \cap V_\alpha$ (because $|X| = \mathfrak{c} \leq \lambda$). In the previous steps $\xi \leq \alpha$, in the same way we had picked in V_ξ a cofinal $\{F_{\xi,\eta}\colon \eta < \lambda\}$ on $[X]^{<\theta} \cap V_\xi$. If $\alpha \notin K$ then we can force with any ccc poset, but when $\alpha \in K$, the book-keeping function h makes the choice: letting $h(\alpha) = (\xi, \eta)$, pick $F_{h(\alpha)} = F_{\xi,\eta}$ (which exists because $\xi \leq \alpha$). As before, let N_α be a transitive model of ZFC such that $F_{h(\alpha)} \subseteq N_\alpha$ and $|N_\alpha| < \theta$, and we force with $\mathbb{Q}_\alpha := \mathbb{Q}_R^{N_\alpha}$ to go to the step $\alpha + 1$.

At the end of the iteration, in V_λ, we have ensured that each member of $\{F_{h(\alpha)}\colon \alpha \in K\} = \{F_{\xi,\eta}\colon \xi, \eta < \lambda\}$ is R-bounded. It remains to ensure that this family is cofinal in $[X]^{<\theta}$: If $F \in [X]^{<\theta}$ then, since $\mathrm{cf}(\lambda) \geq \theta$, we have that $F \in V_\xi$ for some $\xi < \lambda$, so $F \subseteq F_{\xi,\eta}$ for some $\eta < \lambda$.

Concerning $\mathrm{cof}([\lambda]^{<\theta})$, by Fact 1.3 we have that $\mathrm{cf}(\mathrm{cof}([\lambda]^{<\theta})) \geq \mathrm{add}([\lambda]^{<\theta}) = \mathrm{cf}(\theta)$, so $\mathrm{cof}([\lambda]^{<\theta}) = \lambda$ implies $\mathrm{cf}(\lambda) \geq \mathrm{cf}(\theta)$, which is $\mathrm{cf}(\lambda) \geq \theta$ in the case when θ is regular.

Using the book-keeping argument presented above, we are now ready to present the first important construction of models with several pairwise different cardinal characteristics.

Theorem 2.28. *Let $\aleph_1 \leq \theta_1 \leq \theta_2 \leq \theta_3 \leq \theta_4$ be regular cardinals, and assume λ is a cardinal such that $\lambda = \lambda^{\aleph_0}$ and $\mathrm{cf}([\lambda]^{<\theta_i}) = \lambda$ for $i = 1, \ldots, 4$. Then, we can construct a FS iteration of length (and size) λ of ccc posets forcing $\mathrm{add}(\mathcal{N}) = \theta_1$, $\mathrm{cov}(\mathcal{N}) = \theta_2$, $\mathfrak{b} \geq \theta_3$, $\mathrm{non}(\mathcal{E}) = \mathrm{non}(\mathcal{M}) = \theta_4$ and $\mathrm{cov}(\mathcal{M}) = \mathfrak{c} = \lambda$ (see Figure 6).*

Unfortunately, we only deal with $\mathfrak{b} \geq \theta_3$ for the moment. In Section 4 we are going to show how to obtain $\mathfrak{b} = \theta_3$, in addition.

In the first part of the proof we only deal with equalities of the form $\mathfrak{b}(R) \geq \theta_i$ and $\mathfrak{c} = \lambda$. In the next section, we deal with the rest of the proof.

Proof of Theorem 2.28, part 1. Partition $\lambda = K_1 \cup K_2 \cup K_3 \cup K_4$ with $|K_i| = \lambda$. Proceed in two steps:

Step 1. Force with $\mathbb{C}_\lambda$ (i.e. add λ-many Cohen reals).

Step 2. In $V_0 := V^{\mathbb{C}_\lambda}$, using book-keeping as before at each K_i, iterate with length λ and at:

[10]Because, when θ is uncountable, in any ccc generic extension, any set of ordinals of size $<\theta$ is covered by a set in the ground model of size $<\theta$.

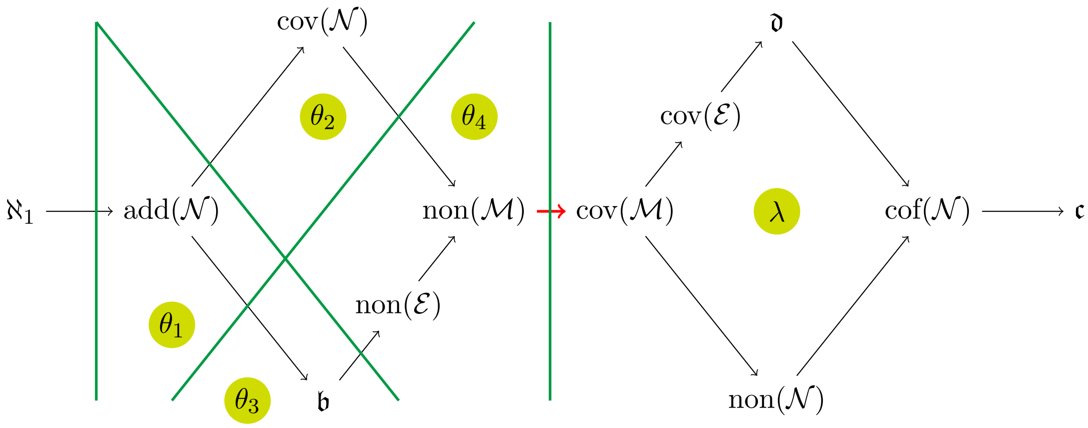

Figure 6: The constellation of Cichoń's diagram forced in Theorem 2.28. For $\mathfrak{b}$ it is only guaranteed that $\mathfrak{b} \leq \theta_3$, but the converse inequality is settled in Section 4.

$\alpha \in K_1$: force with $\mathbb{L}\mathrm{c}_{\mathrm{id}}^{N_\alpha}$,[11] $|N_\alpha| < \theta_1$, which guarantees $\mathrm{add}(\mathcal{N}) \geq \theta_1$ in the final extension;

$\alpha \in K_2$: force with $(\mathcal{B}(2^\omega) \smallsetminus \mathcal{N}(2^\omega))^{N_\alpha}$, $|N_\alpha| < \theta_2$, which guarantees $\mathrm{cov}(\mathcal{N}) \geq \theta_2$ in the final extension;

$\alpha \in K_3$: force with $\mathbb{D}^{N_\alpha}$, $|N_\alpha| < \theta_3$, which guarantees $\mathfrak{b} \geq \theta_3$ in the final extension;

$\alpha \in K_4$: force with $\mathbb{E}^{N_\alpha}$, $|N_\alpha| < \theta_4$, which guarantees $\mathrm{non}(\mathcal{M}) \geq \theta_4$, and even $\mathrm{non}(\mathcal{E}) \geq \theta_4$ (by Theorem 2.15), in the final extension.

It is clear by the construction that, in V_λ, $\mathfrak{c} = \lambda$. □

Note that we have not used the Cohen reals from step 1. These will be used to prove the converse inequalities in the next section.

3 Preservation theory for cardinal characteristics

We deal with the problem of forcing $\mathfrak{b}(R) \leq \theta$ (and much more) to conclude the proof of Theorem 2.28. We proceed in two steps: we first add a strong type of R-unbounded family (using Cohen reals), and then show that this strong unbounded family is not destroyed in the remaining part of the iteration.

The strong type of unbounded family is defined as follows.

Definition 3.1. Let $R = \langle X, Y, \sqsubset \rangle$ be a relation system, and θ an infinite cardinal. We say that $\{x_i \colon i \in I\} \subseteq X$ is a *θ-R-unbounded family* if $|I| \geq \theta$ and $|\{i \in I \colon x_i \sqsubset y\}| < \theta$ for all $y \in Y$.

Although a θ-R-unbounded family is quite large, it has the property that any subset of size θ is R-unbounded, which guarantees $\mathfrak{b}(R) \leq \theta$. But we get much more, as indicated in the following result.

[11] Here id denotes the identity function on ω.

Lemma 3.2. *Assume that $|I| \geq \theta$. Then there exists a θ-R-unbounded family $\{x_i : i \in I\}$ iff $\mathbf{C}_{[I]^{<\theta}} \preceq_{\mathrm{T}} \mathfrak{b}(R)$. In particular, $\mathfrak{b}(R) \leq \mathrm{non}([I]^{<\theta}) = \theta$ and $\mathrm{cov}([I]^{<\theta}) \leq \mathfrak{d}(R)$.*

Proof. If $\{x_i : i \in I\}$ is a θ-R-unbounded family then the maps

$$i \mapsto x_i \text{ and } y \mapsto \{i \in I : x_i \sqsubset y\}$$

yield the desired Tukey connection.

Conversely, assume that $\mathbf{C}_{[I]^{<\theta}} \preceq_{\mathrm{T}} R$ is witnessed by the Tukey connection $i \mapsto x_i$ and $y \mapsto A_y \in [I]^{<\theta}$, i.e. $x_i \sqsubset y$ implies $i \in A_y$. Therefore $\{i \in I : x_i \sqsubset y\} \subseteq A_y$, so it has size $<\theta$. Hence, $\{x_i : i \in I\}$ is a θ-R-unbounded family. □

Concerning $\mathrm{cov}([I]^{<\theta})$, we have

$$\mathrm{cov}([I]^{<\theta}) = \begin{cases} |I| & \text{if } \theta < |I|, \\ \mathrm{cf}(\theta) & \text{if } \theta = |I|. \end{cases}$$

Hence $\mathrm{cov}([I]^{<\theta}) = |I|$ when θ is regular.

We use the following type of relational system of the reals for our θ-unbounded families.

Definition 3.3. We say that $R = \langle X, Y, \sqsubset \rangle$ is a *Polish relational system (Prs)* if the following is satisfied:

(i) X is a perfect Polish space,

(ii) Y is a non-empty analytic subspace of some Polish space Z and

(iii) $\sqsubset = \bigcup_{n<\omega} \sqsubset_n$ where $\langle \sqsubset_n \rangle_{n<\omega}$ is some increasing sequence of closed subsets of $X \times Z$ such that $(\sqsubset_n)^y = \{x \in X : x \sqsubset_n y\}$ is closed nowhere dense for any $n < \omega$ and $y \in Y$.

By (iii), the maps $x \mapsto x$ and $y \mapsto \{x \in X : x \sqsubset y\} \in \mathcal{M}(X)$ form a Tukey connection for $\mathbf{C}_{\mathcal{M}(X)} \preceq_{\mathrm{T}} R$. Moreover:

Fact 3.4. *Any Cohen real $x \in X$ over V is R-unbounded over V.*

Example 3.5. The following relational systems are Polish:

(1) $\mathbf{Lc}(\omega, H_*)$ where $H_* := \{\mathrm{id}^{k+1} : k < \omega\}$ (powers of the identity function on ω), see Definition 2.3.

(2) $\mathbf{Cn}$, see Definition 2.6.

(3) $\langle \omega^\omega, \leq^* \rangle$.

(4) $\mathbf{M}$, see Definition 2.17.

Recall from the previous section that these Polish relational systems describe the cardinal characteristics in Cichoń's diagram, see Figure 7.

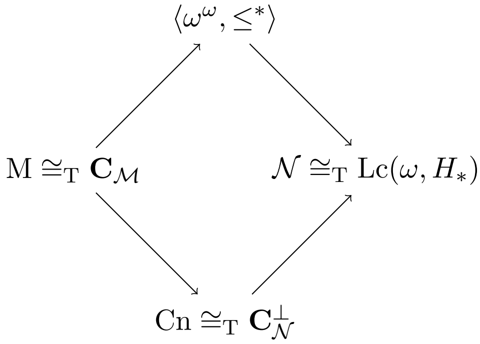

Figure 7: Tukey connections between the relational systems determining the non-dependant values in Cichoń's diagram, along with their equivalent Polish relational systems.

For the rest of this section, we fix a Polish relational system $R = \langle X, Y, \sqsubset \rangle$. In this case, θ-R-unbounded families can easily be added using Cohen reals.

Lemma 3.6. *Let λ be an uncountable cardinal. Then the Cohen reals $\{c_\alpha : \alpha < \lambda\} \subseteq X$ added by $\mathbb{C}_\lambda$ form an $\aleph_1$-R-unbounded family in $V^{\mathbb{C}_\lambda}$.*

Proof. Working in $V^{\mathbb{C}_\lambda}$, let $y \in Y$. Since y is a real, it only depends on countable many maximal antichains, so there is some $C \in [\lambda]^{<\aleph_1} \cap V$ such that $y \in V^{\mathbb{C}_C}$. For any $\alpha \in \lambda \smallsetminus C$, c_α is Cohen over $V^{\mathbb{C}_C}$, hence R-unbounded over $V^{\mathbb{C}_C}$ by Fact 3.4, so $c_\alpha \not\sqsubset y$. Therefore, $\{\alpha < \lambda : c_\alpha \sqsubset y\} \subseteq C$, which is countable. □

Note that $\theta \leq \theta'$ implies that any θ-R-unbounded family $\{x_i : i \in I\}$ is θ'-R-unbounded, as long as $\theta' \leq |I|$. Therefore, the Cohen reals added by $\mathbb{C}_\lambda$ form a θ-R-unbounded family for all $\aleph_1 \leq \theta \leq \lambda$.

The reason we start with $\mathbb{C}_\lambda$ in Step 1 of the proof of Theorem 2.28 is to add θ_i-unbounded families. Now we aim to show how to preserve them in the iteration of Step 2. For this purpose, we introduce the preservation theory from Judah and Shelah [JS90] and Brendle [Bre91].

Definition 3.7. Let κ be an infinite cardinal. A poset $\mathbb{P}$ is *κ-R-good* if, for any $\mathbb{P}$-name $\dot{y}$ for a member of Y, there is a non-empty set $H \subseteq Y$ (in the ground model) of size $<\kappa$ such that, for any $x \in X$, if x is R-unbounded over H then $\Vdash x \not\sqsubset \dot{h}$.

We say that $\mathbb{P}$ is *R-good* if it is $\aleph_1$-R-good.

Note that $\kappa \leq \kappa'$ implies that any κ-R-good poset is κ'-R-good.

Goodness guarantees the preservation of strong unbounded families as follows.

Lemma 3.8. *If κ and θ are infinite cardinals, and $\kappa \leq \mathrm{cf}(\theta)$, then any κ-R-good poset preserves all the θ-R-unbounded families from the ground model.*

Proof. Let $\mathbb{P}$ be a κ-R-good poset. Assume that $\{x_i : i \in I\} \subseteq X$ is a θ-R-unbounded family. Let $\dot{y}$ be a $\mathbb{P}$-name of a member of Y. Find $H \in [Y]^{<\kappa}$ non-empty as in Definition 3.7. For each $y \in H$ let $A^y := \{i \in I : x_i \sqsubset y\}$ and $A := \bigcup_{y \in H} A^y$. Then $|A^y| < \theta$ and $|A| < \theta$, the latter because $\mathrm{cf}(\theta) \geq \kappa$.

We claim that $\Vdash \{i \in I : x_i \sqsubset y\} \subseteq A$. Indeed, if $i \in I \smallsetminus A$, $x_i \not\sqsubset y$ for all $y \in H$, so $\Vdash x_i \not\sqsubset \dot{y}$. □

Now, goodness is preserved along FS iterations.

Theorem 3.9. *Let κ be an uncountable regular cardinal. Then, any FS iteration of κ-cc κ-R-good posets is again κ-R-good.*

Proof. See e.g. [CM19, Thm. 4.15]. □

This result can be weakened as follows.

Theorem 3.10. *Let κ and θ be uncountable cardinals such that κ is regular and $\mathrm{cf}(\theta) \geq \kappa$. Then, any FS iteration of κ-cc posets preserving θ-R-unbounded families, preserves θ-R-unbounded families.*

We now turn to particular cases. One very useful fact is that small posets are good.

Lemma 3.11. *Any poset $\mathbb{P}$ is κ-R-good for any infinite $\kappa > |\mathbb{P}|$.*

In particular, Cohen forcing is κ-R-good for all uncountable κ.

Proof. See [Mej13, Lem. 4], also [CM19, Lem. 4.10] □

More concrete examples of R-good posets comes from the connection between the combinatorics of a forcing with R. We formalize this with the following notions.

Definition 3.12 ([Mej19])**.** We say that Γ is a *linkedness property* if $\Gamma(\mathbb{P}) \subseteq \mathcal{P}(\mathbb{P})$ for any poset $\mathbb{P}$.

Let μ and κ be infinite cardinals.

(1) A poset $\mathbb{P}$ is *μ-Γ-linked* if it can be covered by $\leq\mu$-many subsets in $\Gamma(\mathbb{P})$.

When $\mu = \aleph_0$, we write *σ-Γ-linked*.

(2) A poset $\mathbb{P}$ is *κ-Γ-Knaster* if $\forall B \in [\mathbb{P}]^\kappa\ \exists A \in [B]^\kappa \colon A \in \Gamma(\mathbb{P})$.

When $\kappa = \aleph_1$, we just write *Γ-Knaster*.

If Γ satisfies that $Q' \subseteq Q \in \Gamma(\mathbb{P})$ implies $Q' \in \Gamma(\mathbb{P})$, then any μ-Γ-linked poset is μ^+-Γ-Knaster. A more concrete discussion about linkedness properties and iterations can be found in [Mej19, Sec. 5].

Example 3.13. The following are examples of linkedness properties. Here, $\mathbb{P}$ denotes an arbitrary poset.

(1) $\Lambda_{<\omega}$: *Centered.* $Q \in \Lambda_{<\omega}(\mathbb{P})$ iff Q is a centered subset of $\mathbb{P}$, i.e. for any finite $F \subseteq Q$, there is a $q \in \mathbb{P}$ stronger that all members of $\mathbb{P}$.

Then, μ-$\Lambda_{<\omega}$-linked means μ-centered, and κ-$\Lambda_{<\omega}$-Knaster means precaliber κ.

(2) Λ_{int}: *Positive intersection number.* For $n<\omega$ non-zero and $s\in\mathbb{P}^n$, define

$$\iota_*(s) := \max\{|e| \colon e\subseteq n \text{ and } \{s_i\colon i\in e\} \text{ has a common lower bound in } \mathbb{P}\}.$$

For $Q\subseteq\mathbb{P}$, define *the intersection number of Q in $\mathbb{P}$* by

$$\operatorname{int}_{\mathbb{P}}(Q) := \inf\left\{\frac{\iota_*(s)}{n}\colon s\in Q^n,\ 0<n<\omega\right\}.$$

We say that $Q\in\Lambda_{\text{int}}$ iff $\operatorname{int}_{\mathbb{P}}(Q)>0$.

Notice that $\Lambda_{<\omega}(\mathbb{P})\subseteq\Lambda_{\text{int}}(\mathbb{P})$ because any centered poset has intersection number 1.

According to the following result, $\Lambda_{<\omega}$ *is good for* Cn:

Theorem 3.14 (Brendle [Bre91])**.** *Any μ-centered poset is μ^+-*Cn*-good. In particular, any σ-centered poset is* Cn*-good.*

Inspired by a result of Kamburelis [Kam89], we have that Λ_{int} *is good for* $\mathrm{Lc}(\omega,H_*)$. Recall that $H_*=\{\mathrm{id}^{k+1}\colon k<\omega\}$.

Theorem 3.15. *Any μ-Λ_{int}-linked poset if μ^+-$\mathrm{Lc}(\omega,H_*)$-good.*

Corollary 3.16. *Any μ-centered poset is μ^+-$\mathrm{Lc}(\omega,H_*)$-good.*

Other examples are obtained using Boolean algebras with finitely additive measures.

Theorem 3.17 (Kelley [Kel59])**.** *Let $\mathbb{B}$ be a Boolean algebra. Then $\mathbb{B}\smallsetminus\{0_{\mathbb{B}}\}$ is σ-Λ_{int}-linked iff there is a strictly positive fam $\Xi\colon\mathbb{B}\to[0,1]$ (i.e. $\Xi(b)=0$ iff $b=0$).*

In combination with Theorem 3.15, we obtain

Corollary 3.18. *If N is a transitive model of* ZFC*, then $(\mathcal{B}(2^\omega)\smallsetminus\mathcal{N}(2^\omega))^N$ is $\mathrm{Lc}(\omega,H_*)$-good.*

In the next section, we will present a good linkedness property for $\langle\omega^\omega,\leq^*\rangle$. For the moment, we present the following examples.

Theorem 3.19 (Miller [Mil81])**.** *$\mathbb{E}$ is $\langle\omega^\omega,\leq^*\rangle$-good.*

Theorem 3.20. *Random forcing is $\langle\omega^\omega,\leq^*\rangle$-good.*[12]

We are finally ready to conclude the proof of Theorem 2.28.

Proof of Theorem 2.28, part 2. It remains to show that, in V_λ, $\operatorname{add}(\mathcal{N})\leq\theta_1$, $\operatorname{cov}(\mathcal{N})\leq\theta_2$, $\operatorname{non}(\mathcal{M})\leq\theta_4$ and $\lambda\geq\operatorname{cov}(\mathcal{M})$.

By Lemma 3.6, the Cohen reals added at step 1 gives us $\aleph_1$-R-unbounded families of size λ for any Polish relational system R, in particular, we obtain in $V^{\mathbb{C}_\lambda}$ an $\aleph_1$-$\mathrm{Lc}(\omega,H_*)$-unbounded $\{c^1_\alpha\colon\alpha<\lambda\}$, an $\aleph_1$-Cn-unbounded $\{c^2_\alpha\colon\alpha<\lambda\}$, and an $\aleph_1$-M-unbounded

[12]This easily follows from the fact that random forcing is ω^ω-bounding.

$\{c^4_\alpha \colon \alpha < \lambda\}$. Now, if we prove that the iteration of step 2 is θ_1-$\mathrm{Lc}(\omega, H_*)$-good, θ_2-Cn-good and θ_4-M-good, we obtain by Lemma 3.8 that the previous families are, in the final extension, θ_1-$\mathrm{Lc}(\omega, H_*)$-unbounded, θ_2-Cn-unbounded, and θ_4-M-unbounded, respectively. Therefore, by Lemma 3.2, $\mathrm{add}(\mathcal{N}) = \mathfrak{b}(\mathrm{Lc}(\omega, H_*)) \leq \theta_1$, $\mathrm{cof}(\mathcal{N}) = \mathfrak{b}(\mathrm{Cn}) \leq \theta_2$, $\mathrm{non}(\mathcal{M}) = \mathfrak{b}(\mathrm{M}) \leq \theta_4$ and $\mathrm{cov}(\mathcal{M}) = \mathfrak{d}(\mathrm{M}) \geq \mathrm{cov}([\lambda]^{<\theta_4}) = \lambda$.

By virtue of Theorem 3.9, it is enough to prove that all the iterands used in step 2 are θ_1-$\mathrm{Lc}(\omega, H_*)$-good, θ_2-Cn-good and θ_4-M-good. Indeed, for:

$\alpha \in K_1$: $\mathbb{Q}_\alpha = \mathbb{L}\mathrm{c}^{N_\alpha}_{\mathrm{id}}$ has size $< \theta_1$ because $|N_\alpha| < \theta_1$, so it is θ_1-R-good (and κ-R-good for any $\kappa \geq \theta_1$) for any Polish relational system R (by Lemma 3.11).

$\alpha \in K_2$: $\mathbb{Q}_\alpha = (\mathcal{B}(2^\omega)/\mathcal{N}(2^\omega))^{N_\alpha}$ has size $<\theta_2$, so it is θ_2-R-good for any Polish relational system R. On the other hand, by Corollary 3.18, $\mathbb{Q}_\alpha$ is $\mathrm{Lc}(\omega, H_*)$-good.

$\alpha \in K_3$: $\mathbb{Q}_\alpha = \mathbb{D}^{N_\alpha}$ has size $<\theta_3$, so it is θ_3-R-good for any Polish relational system R. On the other hand, $\mathbb{Q}_\alpha$ is $\mathrm{Lc}(\omega, H_*)$-good and Cn-good by Theorem 3.14 and Corollary 3.16, respectively.

$\alpha \in K_4$: $\mathbb{Q}_\alpha = \mathbb{E}^{N_\alpha}$ has size $<\theta_4$, so it is θ_4-R-good for any Polish relational system R. On the other hand, $\mathbb{Q}_\alpha$ is $\mathrm{Lc}(\omega, H_*)$-good and Cn-good by Theorem 3.14 and Corollary 3.16, respectively. □

In the previous proof, we have that $\mathbb{Q}_\alpha$ is θ_3-$\langle \omega^\omega, \leq^* \rangle$-good for $\alpha \in K_1 \cup K_2 \cup K_3$. However, although $\mathbb{E}$ is $\langle \omega^\omega, \leq^* \rangle$-good, it is unclear whether restrictions of the form $\mathbb{E}^N$ for transitive models N of ZFC are $\langle \omega^\omega, \leq^* \rangle$-good. There are counter-examples when N is a proper-class model:

Theorem 3.21 (Pawlikowsi [Paw92]). *There is a proper-ω^ω-bounding generic extension W of V in where $\mathbb{E}^V$ and $(\mathcal{B}(2^\omega) \smallsetminus \mathcal{N}(2^\omega))^V$ add dominating reals over W.*

Even more, in the case of random forcing:

Theorem 3.22 (Judah and Shelah [JS93]). *There is a ccc forcing extension W of V such that $(\mathcal{B}(2^\omega) \smallsetminus \mathcal{N}(2^\omega))^V$ adds dominating reals over W.*

In the next section, we modify the forcing construction of Theorem 2.28 for $\alpha \in K_4$ to guarantee that the set of Cohen reals $\{c^3_\alpha \colon \alpha < \lambda\}$ added by $\mathbb{C}_\lambda$ stays θ_3-$\langle \omega^\omega, \leq^* \rangle$-unbounded in the final extension.

4 FS iterations with measures and ultrafilters on the natural numbers

We show how to modify the iteration in Theorem 2.28 to force, in addition, $\mathfrak{b} \leq \theta_3$. We start by introducing the following good property for $\langle \omega^\omega, \leq^* \rangle$.

Definition 4.1 ([Mej19, BCM21])**.** Let $F \subseteq \mathcal{P}(\omega)$ be a filter. We assume that all filters are *free*, i.e. they contain the *Frechet filter* $\mathrm{Fr} := \{\omega \smallsetminus a \colon a \in [\omega]^{<\aleph_0}\}$. A set $a \subseteq \omega$ is *F-positive* if it intersects every member of F. Denote by F^+ the collection of F-positive sets.

We define the linkedness property Λ_F, which we call *F-linked*: given a poset $\mathbb{P}$ and $Q \subseteq \mathbb{P}$, Q is F-linked if, for any $\langle p_n \colon n < \omega\rangle \in Q^\omega$, there is some $q \in \mathbb{P}$ such that

$$q \Vdash \{n < \omega \colon p_n \in \dot{G}\} \in F^+.$$

Note that, in the case $F = \mathrm{Fr}$, the previous equation is "$q \Vdash \{n < \omega \colon p_n \in \dot{G}\}$ is infinite".

We also define Λ_{uf}, which we call *uf-linked* (*ultrafilter-linked*): $Q \in \Lambda_{\mathrm{uf}}(\mathbb{P})$ if $Q \in \Lambda_F(\mathbb{P})$ for every (ultra)filter F on ω.

If F and F' are filters on ω, it is clear that $\Lambda_{\mathrm{uf}}(\mathbb{P}) \subseteq \Lambda_{F'}(\mathbb{P}) \subseteq \Lambda_F(\mathbb{P}) \subseteq \Lambda_{\mathrm{Fr}}(\mathbb{P})$. But, for ccc posets:

Lemma 4.2 ([Mej19])**.** *If $\mathbb{P}$ is ccc then $\Lambda_{\mathrm{uf}}(\mathbb{P}) = \Lambda_{\mathrm{Fr}}(\mathbb{P})$.*

Example 4.3.

(1) Any singleton is uf-linked. Hence, any poset $\mathbb{P}$ is $|\mathbb{P}|$-uf-linked. In particular, Cohen forcing is σ-uf-linked.

(2) Random forcing is σ-uf-linked, in fact, any measure algebra is σ-uf-linked. Indeed, if $\mathbb{B}$ is a complete Boolean algebra and $\mu \colon \mathbb{B} \to [0,1]$ is a σ-additive measure such that $\mu(p) \neq 0$ for all $p \neq 0_{\mathbb{B}}$, then, for any $\delta > 0$, $\{p \in \mathbb{B} \colon \mu(p) \geq \delta\}$ is Fr-linked.

(3) The forcing $\mathbb{E}$ (see Definition 2.14) is σ-uf-linked. We show later that this poset satisfies a stronger property.

The following series of results indicate that Λ_{Fr} is good for $\langle \omega^\omega, \leq^* \rangle$.

Lemma 4.4. *Let $\mathbb{P}$ be a poset and $Q \subseteq \mathbb{P}$. Then Q is* Fr*-linked iff, for any $\mathbb{P}$-name $\dot{m}$ for a natural number, there is some $m' \in \omega$ (in the ground model) such that no $p \in Q$ forces $m' \leq \dot{m}$.*

Proof. ($\Rightarrow$) [Mej19] Assume that, for any $n < \omega$, there is some $p_n \in Q$ forcing $n \leq \dot{m}$. Then, if G is $\mathbb{P}$-generic over V, then $\{n < \omega \colon p_n \in G\}$ must be finite because $p_n \in G \Rightarrow n \leq \dot{m}[G] < \omega$. Therefore, in V, Q cannot be Fr-linked.

($\Leftarrow$) (with Cardona) Assume that Q is not Fr-linked, so there is some $\langle p_n \colon n < \omega\rangle \in Q^\omega$ such that $\Vdash$"$\{n < \omega \colon p_n \in \dot{G}\}$ is finite". So pick some $\mathbb{P}$-name $\dot{m}$ of a natural number such that $\Vdash$"$\{n < \omega \colon p_n \in \dot{G}\} \subseteq \dot{m}$. Note that $p_n \Vdash n < \dot{m}$. ☐

Lemma 4.5. *Let $\mathbb{P}$ be a poset and Q be an* Fr*-linked subset of $\mathbb{P}$. If $\dot{y}$ is a $\mathbb{P}$-name of a member of ω^ω, then there is some $y' \in \omega^\omega$ (in the ground model) such that, for any $x \in \omega^\omega$*

$$x \not\leq^* y' \Rightarrow \forall n < \omega\ \forall p \in Q \colon p \nVdash \forall i \geq n \colon x(k) \leq \dot{y}(k).$$

Proof. Using Lemma 4.4, for each $k < \omega$ find $y'(k) < \omega$ such that no $p \in Q$ forces $y'(k) \leq \dot{y}(k)$. This defines $y' \in \omega^\omega$.

Now assume that $x \in \omega^\omega$ and $x \not\leq^* y'$. Let $n < \omega$ and $p \in Q$, so there is some $k \geq n$ such that $x(k) > y'(k)$. On the other hand, $p \not\Vdash y'(k) \leq \dot{y}(k)$, so there is some $q \leq p$ forcing $\dot{y}(k) < y'(k) < x(k)$, so $p \not\Vdash \forall\, k \geq n\colon x(k) \leq \dot{y}(k)$. □

Theorem 4.6 ([Mej19]). *Any μ-*Fr*-linked poset is μ^+-ω^ω-good.*

This theorem is an easy consequence of Lemma 4.5. However, we do not know how to modify the construction in Theorem 2.28 to obtain a θ_3-ω^ω-good iteration. But we have some other way to preserve unbounded families, as in the following result.

Theorem 4.7 ([BCM21]). *Let θ be an uncountable regular cardinal. Then any θ-*Fr*-Knaster poset preserves θ-ω^ω-unbounded families.*

Proof. Assume that $\{x_i\colon i \in I\}$ is a θ-ω^ω-unbounded family, and that there is some $p \in \mathbb{P}$ forcing that it is not, i.e. for some $\mathbb{P}$-name $\dot{y}$ of a member of ω^ω, p forces that $|\{i \in I\colon x_i \leq^* \dot{y}\}| \geq \theta$. This implies that the set

$$I_0 := \{i \in I\colon \exists\, p' \leq p\colon p' \Vdash x_i \leq^* \dot{y}\}$$

has size $\geq\theta$. Pick $I_1 \subseteq I_0$ of size θ and, for each $i \in I_1$, choose $p_i \leq p$ and $n_i < \omega$ such that $p_i \Vdash x_i(k) \leq \dot{y}(k)$ for all $k \geq n_i$. Since $\mathrm{cf}(\theta) > \omega$, we can find $n < \omega$ and $I_2 \subseteq I_1$ of size θ such that $n_i = n$ for all $i \in I_2$.

Since θ is regular and $\mathbb{P}$ is θ-Fr-Knaster, there is some $I' \subseteq I_2$ such that the set $Q := \{p_i\colon i \in I'\}$ is Fr-linked. Now find $y' \in \omega^\omega$ as in Lemma 4.5 for $\dot{y}$ and Q. Then, $|\{i \in I\colon x_i \leq^* y'\}| < \theta$, so there is some $i \in I'$ such that $x_i \not\leq^* y'$. Hence, by Lemma 4.5, no $p \in Q$ forces $\forall\, k \geq n\colon x_i(k) \leq \dot{y}(k)$. But p_i forces this, a contradiction. □

Because of the previous theorem, the plan now is to modify the construction of Theorem 2.28 to obtain a θ_3-Fr-Knaster poset. To achieve this, we use the following linkedness property, stronger than ultrafilter-linkedness.

Definition 4.8 (cf. [GMS16]). Given a (non-principal) ultrafilter D on ω, define the linkedness property $\Lambda^{\lim}_D$, called D-lim-*linked*: $Q \in \Lambda^{\lim}_D(\mathbb{P})$ if there are a $\mathbb{P}$-name $\dot{D}'$ of an ultrafilter on ω extending D and a map $\lim^D\colon Q^\omega \to \mathbb{P}$ such that, whenever $\bar{p} = \langle p_n\colon n < \omega\rangle \in Q^\omega$,

$$\lim{}^D \bar{p} \Vdash \{n < \omega\colon p_n \in \dot{G}\} \in \dot{D}'.$$

Define the linkedness property $\Lambda^{\lim}_{\mathrm{uf}}$, called *uf*-lim-*linked*, by $Q \in \Lambda^{\lim}_{\mathrm{uf}}(\mathbb{P})$ iff $Q \in \Lambda^{\lim}_D(\mathbb{P})$ for any ultrafilter D on ω.

In addition, for an infinite cardinal μ, we say that a poset $\mathbb{P}$ is *uniformly μ-D-*lim*-linked* if if is μ-$\Lambda^{\lim}_D$-linked witnessed by some $\langle Q_\alpha\colon \alpha < \mu\rangle$, but the $\dot{D}'$ above can be the same for any Q_α. And we say that $\mathbb{P}$ is *uniformly μ-uf-*lim*-linked* if there is some $\langle Q_\alpha\colon \alpha < \mu\rangle$ witnessing that $\mathbb{P}$ is uniformly μ-D-lim-linked for any ultrafilter D on ω.

Example 4.9. Any singleton is uf-lim-linked. As a consequence, any poset $\mathbb{P}$ is uniformly $|\mathbb{P}|$-uf-lim-linked, witnessed by its singletons: for $p \in P$, let $Q_p := \{p\}$, and $\lim^D$ on Q_p is just the constant map with value p, when D is an ultrafilter on ω. Since $\lim^D \bar{p} \Vdash \{n < \omega\colon p_n \in \dot{G}\} = \omega$ for all $\bar{p} \in Q_p^\omega$, $\dot{D}'$ can be any $\mathbb{P}$-name of an ultrafilter extending D.

Theorem 4.10 ([GMS16, BCM21]). *$\mathbb{E}$ is uniformly σ-uf-*lim*-linked.*

Proof. We only indicate the components and the limit functions. For $s \in \omega^{<\omega}$ and $m \in \omega$, consider the set $E_{s,m}$ of conditions in $\mathbb{E}$ of the form (s, φ) with $\varphi \in \mathcal{S}(\omega, m)$. If D is an ultrafilter on ω and $\bar{p} = \langle p_n \colon n < \omega \rangle \in E^{\omega}_{s,m}$, $p_n = (s, \varphi_n)$, define $\lim^D \bar{p} := (s, \varphi)$ where

$$k \in \varphi(i) \text{ iff } \{n < \omega \colon k \in \ \varphi_n(i)\} \in D.$$

It is clear that $(s, \varphi) \in E_{s,m}$.

The sequence $\langle E_{s,m} \colon s \in \omega^{<\omega},\ m < \omega \rangle$ witnesses that $\mathbb{E}$ is uniformly σ-D-lim-linked for any ultrafilter D on ω. This is proved by showing that, whenever G is $\mathbb{P}$-generic over V, the set

$$D \cup \bigcup_{s,m} \{\{n < \omega \colon p_n \in G\} \colon \bar{p} \in E^{\omega}_{s,m} \cap V,\ \lim^D p \in G\}$$

has the finite intersection property. □

We present a framework to construct FS iterations that allow ultrafilter limits. The candidates for such iterations can be presented in a more general fashion. For an infinite cardinal θ, denote

$$\theta^- = \begin{cases} \theta_0 & \text{if } \theta = \theta_0^+ \text{ for some cardinal } \theta_0, \\ \theta & \text{if } \theta \text{ is not a successor cardinal.} \end{cases}$$

Definition 4.11. Let θ be an uncountable cardinal. A FS iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\xi \colon \alpha \leq \pi,\ \xi < \pi \rangle$ is a *θ-Γ-iteration* if it satisfies:

(i) $\mathbb{P}^-_\xi \lessdot \mathbb{P}_\xi$ for all $\xi < \pi$, and

(ii) $\mathbb{P}^-_\xi$ forces that $\dot{\mathbb{Q}}_\xi$ is μ_ξ-Γ-linked witnessed by a sequence of $\mathbb{P}^-_\xi$-names $\langle \dot{Q}_{\xi,\zeta} \colon \zeta < \mu_\xi \rangle$, where $\mu_\xi < \theta$ (known from the ground model).

Associated with this iteration, we define the following notions.

(1) A function $h \colon d_h \to \theta^-$ with $\pi \subseteq d_h$ is usually called a *guardrail for the iteration.*

(2) For $\alpha \leq \pi$ and h as above, let $\mathbb{P}^h_\alpha$ be the set of conditions $p \in \mathbb{P}_\alpha$ *following* h, i.e. for $\xi \in \operatorname{dom} p$, $h(\xi) < \mu_\xi$, $p(\xi)$ is a $\mathbb{P}^-_\xi$-name and $\Vdash_{\mathbb{P}^-_\xi} p(\xi) \in \dot{Q}_{\xi,h(\xi)}$.

(3) $\mathbb{P}^*_\alpha := \bigcup_{h \in \theta^{-\pi}} \mathbb{P}^h_\alpha$.

(4) Let L be a linear order and $\langle p_\ell \colon \ell \in L \rangle$ a sequence of conditions in $\mathbb{P}_\pi$. We say that $\langle p_\ell \colon \ell \in L \rangle$ is a *uniform Δ-system* if it satisfies the following:

 (i) All $\operatorname{dom} p_\ell$ ($\ell \in L$) have the same size n: $\operatorname{dom} p_\ell = \{\alpha_{\ell,k} \colon k < n\}$ (increasing enumeration).

 (ii) There is some $v \subseteq n$ such that, for each $k \in v$, the sequence $\langle \alpha_{\ell,k} \colon \ell \in L \rangle$ is constant with value $\alpha_{*,k}$.

 (iii) $\langle \operatorname{dom} p_\ell \colon \ell \in L \rangle$ forms a Δ-system with root $\{\alpha_{*,k} \colon k \in v\}$.

 (iv) For $k \in n \smallsetminus v$, the sequence $\langle \alpha_{\ell,k} \colon \ell \in L \rangle$ is increasing.

(v) There is some guardrail h such that $\{p_\ell \colon \ell \in L\} \subseteq \mathbb{P}^h_\pi$.

By recursion on $\alpha \leq \pi$, we can show:

Fact 4.12. *For any θ-Γ-iteration as in Definition 4.11, $\mathbb{P}^*_\alpha$ is dense in $\mathbb{P}_\alpha$.*

We focus on the case $\Gamma = \Lambda^{\mathrm{lim}}_{\mathrm{uf}}$. We plan to construct a θ-$\Lambda^{\mathrm{lim}}_{\mathrm{uf}}$-iteration which is θ-Fr-Knaster (in our case, $\theta = \theta_3$).

Lemma 4.13. *For a θ-Λ_{Fr}-iteration as in Definition 4.11: Let H be a set of guardrails, $\theta' \geq \theta$ regular, and assume:*

(i) *Any countable partial function from π into θ^- can be extended by some $h \in H$.*

(ii) *If $h \in H$ and $\bar{p} = \langle p_n \colon n < \omega\rangle \subseteq \mathbb{P}^h_\pi$ forms a uniform Δ-system, then there is some $q \in \mathbb{P}_\pi$ forcing that $\{n < \omega \colon p_n \in \dot{G}\}$ is infinite.*

*Then $\mathbb{P}_\pi$ is θ'-*Fr*-Knaster.*

Proof. Let $A \subseteq \mathbb{P}_\pi$ have size θ'. Since θ' is regular uncountable, we can find an uniform Δ-system $B \subseteq A$ of size θ'. Condition (ii) implies that B is Fr-linked. □

The q in (ii) is found as an *ultrafilter limit* similar to Definition 4.8, so this requires to construct ultrafilters along the iteration. For the successor step, the following lemma is useful.

Lemma 4.14 ([BCM21, Lem. 3.20]). *Let $M \subseteq N$ be transitive models of* ZFC *and $\mathbb{Q} \in M$ be a poset. Assume that $M \models$ "D^- is an ultrafilter on ω", $M \models$ "$\dot{D}^+$ is a $\mathbb{Q}$-name of an ultrafilter on ω extending D^-", and $N \models$ "D is an ultrafilter on ω extending D^-". Then, in N, $\mathbb{Q}$ forces that $D \cup D^+$ has the finite intersection property, i.e. it can be extended to an ultrafilter (see Figure 8).*

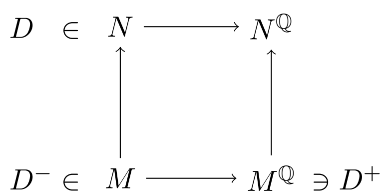

Figure 8: The situation in Lemma 4.14

Definition 4.15. A θ-$\Lambda^{\mathrm{lim}}_{\mathrm{uf}}$-iteration as in Definition 4.11 *has ultrafilter limits for H* when:

(i) H is a set of guardrails,

(ii) for $h \in H$, $\langle \dot{D}^h_\xi \colon \xi \leq \pi\rangle$ is a sequence such that $\dot{D}^h_\xi$ is a $\mathbb{P}_\xi$ name of a non-principal ultrafilter on ω,

(iii) if $\xi < \eta \leq \pi$ then $\Vdash_{\mathbb{P}_\eta} \dot{D}^h_\xi \subseteq \dot{D}^h_\eta$,

(iv) $\mathbb{P}_\xi$ forces that $\dot{D}^h_\xi \cap V^{\mathbb{P}^-_\xi} \in V^{\mathbb{P}^-_\xi}$, and whenever $h \in H$, $\langle \xi_n \colon n < \omega \rangle \subseteq \pi$ and $\Vdash_{\mathbb{P}^-_{\xi_n}} \dot{q}_n \in \dot{Q}_{\xi_n, h(\xi_n)}$:

(v) if $\langle \xi_n \colon n < \omega \rangle$ is constant with value ξ then

$$\Vdash_{\mathbb{P}_\xi} \lim_n^{\dot{D}^h_\xi} \dot{q}_n \Vdash_{\dot{\mathbb{Q}}_\xi} \{n < \omega \colon \dot{q}_n \in \dot{G}(\xi)\} \in \dot{D}^h_{\xi+1},$$

(vi) and if $\langle \xi_n \colon n < \omega \rangle$ is increasing, then

$$\Vdash_{\mathbb{P}_\pi} \{n < \omega \colon \dot{q}_n \in \dot{G}(\xi_n)\} \in \dot{D}^h_\pi$$

Lemma 4.16. *Any iteration as in Definition 4.15 satisfies (ii) of Lemma 4.13 for H.*

Proof. Let $\langle p_n \colon n < \omega \rangle$ be an uniform Δ-system in $\mathbb{P}^h_\pi$. Let Δ be the root of the Δ-system and define $q \in \mathbb{P}_\pi$ with $\operatorname{dom} q := \Delta$ such that $q(\xi)$ is a $\mathbb{P}^-_\xi$-name of $\lim_n^{\dot{D}^h_\xi} p_n(\xi)$ for $\xi \in \Delta$. Then q forces that $\{n < \omega \colon p_n \in \dot{G}_\pi\} \in \dot{D}^h_\pi$. □

To obtain (i) of Definition 4.15 we could basically use $H = \theta^{-\pi}$. However, there are steps $\xi < \pi$ of the iteration where we want $\mathbb{P}^-_\xi$ to be quite small, so to guarantee (iv) of Definition 4.15 we need that H is also small. This is guaranteed by the following result.

Theorem 4.17 ([EK65, Rin12]). *Let ν, κ be infinite cardinals and L be a set such that $\nu \leq \kappa \leq |L| \leq 2^\kappa$. Then there exists an $H \subseteq {}^L\kappa$ such that $|H| \leq \kappa^{<\nu}$, and any partial function from L into κ with domain of size $<\nu$ can be extended by a function in H.*

The following two theorems indicate how to construct iterations as in Definition 4.15.

Theorem 4.18. *Let $\mathbb{P}_{\pi+1}$ be a θ-$\Lambda^{\lim}_{\mathrm{uf}}$-iteration of length $\pi+1$ and H be a set of guardrails such that, up to π, it has ultrafilter limits for H.*

Assume that $\mathbb{P}^-_\pi \subsetdot \mathbb{P}_\pi$ and $\mathbb{P}_\pi$ forces $\dot{D}^h_\pi \cap V^{\mathbb{P}^-_\pi} \in V^{\mathbb{P}^-_\pi}$. Then, we can find $\mathbb{P}_{\pi+1}$-names $\dot{D}^h_{\pi+1}$ ($h \in H$) of ultrafilters extending $\dot{D}^h_\pi$ which make $\mathbb{P}_{\pi+1}$ have ultrafilter limits for H.

Proof. Direct application of Lemma 4.14. □

Theorem 4.19. *Assume that π is a limit ordinal and $\mathbb{P}_\pi$ is a θ-$\Lambda^{\lim}_{\mathrm{uf}}$-iteration of length π. Further assume that h is a guardrail and $\langle \dot{D}^h_\xi \colon \xi < \pi \rangle$ is a sequence witnessing that, for any $\xi < \pi$, $\mathbb{P}_\xi$ is an iteration with uf-limits for h.*

If, for any $\xi < \pi$, $\mathbb{P}^-_\xi$ forces that $Q_{\xi,h(\xi)}$ is centered, then we can find a $\dot{D}^h_\pi$ that makes $\mathbb{P}_\pi$ have uf-limits for h.

We are now ready to present the main forcing construction of this section.

Theorem 4.20 (cf. [GMS16, GKS19]). *Let $\aleph_1 \leq \theta_1 \leq \theta_2 \leq \theta_3 \leq \theta_4$ be regular cardinals, and assume λ is a cardinal such that $\lambda = \lambda^{\aleph_0}$ and $\operatorname{cf}([\lambda]^{<\theta_i}) = \lambda$ for $i = 1, \ldots, 4$. Further assume that one of the following holds:*

(i) $\theta_3 = \theta_4$.

(ii) $\theta_3^- < \theta_4$, $\theta^{\aleph_0} < \theta_4$ *for every cardinal* $\theta < \theta_4$, *and* $\lambda \leq 2^\kappa$ *for some cardinal* $\kappa < \theta_4$.

Then, we can construct a FS iteration of length (and size) λ of ccc posets forcing $\mathrm{add}(\mathcal{N}) = \theta_1$*,* $\mathrm{cov}(\mathcal{N}) = \theta_2$*,* $\mathfrak{b} = \theta_3$*,* $\mathrm{non}(\mathcal{E}) = \mathrm{non}(\mathcal{M}) = \theta_4$ *and* $\mathrm{cov}(\mathcal{M}) = \mathfrak{c} = \lambda$ *(see Figure 6).*

Proof. In case (i) the result follows directly from Theorem 2.28, so we focus on the assumptions of case (ii), in which we can further assume that $\kappa \geq \theta_3^-$. We proceed exactly as in the proof of Theorem 2.28 to construct a FS iteration of length $\pi := \lambda + \lambda$, using Cohen forcing at the first λ stages, but we modify the construction for $\alpha \in K_4$, the steps where we increase $\mathrm{non}(\mathcal{M})$ (and even $\mathrm{non}(\mathcal{E})$) using $\mathbb{E}$, to obtain a θ_3-$\Lambda^{\mathrm{lim}}_{\mathrm{uf}}$-iteration with ultrafilter limits on some H of size $<\theta_4$. We aim to apply Lemma 4.13 and 4.16 to conclude that the iteration is θ_3-Fr-Knaster, hence ensuring that the first λ-many Cohen reals added in the iteration form a θ_3-ω^ω-unbounded family in the final extension, so the remaining $\mathfrak{b} \leq \theta_3$ will be forced.

Using that $\theta_3^- \leq \kappa < \theta_4$ and $\lambda \leq 2^\kappa$, by Theorem 4.17 we can find $H_0 \subseteq \kappa^\pi$ of size $\leq \kappa^{\aleph_0} < \theta_4$ (by (ii)) such that any countable partial function from π into κ can be extended by a function in H_0. For any $g \in \kappa^\pi$ define $g' \in {\theta_3^-}^\pi$ by $g'(\xi) := g(\xi)$ if $g(\xi) < \theta_3^-$, and $g'(\xi) := 0$ otherwise. Then $H := \{g' \colon g \in H_0\} \subseteq {\theta_3^-}^\pi$ has size $<\theta_4$ and any countable partial function from π into θ_3^- can be extended by a function in H. This guarantees requirement (i) of Lemma 4.13.

To construct the iteration, proceed by recursion, starting with an ultrafilter D_0^h on ω for $h \in H$. In the successor step $\xi \to \xi + 1$, we do some work in the case $\xi = \lambda + \alpha$ with $\alpha \in K_4$ because in other cases we proceed as in Theorem 2.28 and just pick $\mu_\xi < \theta_3$ such that $\Vdash_{\mathbb{P}_\xi} \dot{\mathbb{Q}}_\xi = \{\dot{q}^\xi_\zeta \colon \zeta < \mu_\xi\}$, so we let $\mathbb{P}_\xi^- := \mathbb{P}_\xi$ and $\dot{Q}_{\xi,\zeta}$ be a $\mathbb{P}_\xi$-name of $\{\dot{q}^\xi_\zeta\}$, so any $\mathbb{P}_{\xi+1}$-name $\dot{D}^h_{\xi+1}$ of an ultrafilter extending $\dot{D}^h_\xi$ is suitable.

Using the book-keeping for K_4, in stage $\xi = \lambda + \alpha$ we have picked some $\mathbb{P}_\xi$-name $\dot{F}_\alpha$ of a subset of ω^ω of size $<\theta_4$, and aim to add an eventually different real over $\dot{F}_\alpha$ in the following step by using a restriction of $\mathbb{E}$. Since $\mathbb{P}_\xi$ has the ccc, we can find some $\nu_\alpha < \theta_4$ such that $\dot{F}_\alpha$ is represented by $\{\dot{x}_{\alpha,i} \colon i < \nu_\alpha\}$. Using the assumption (ii), for large enough χ we can find $M \prec H_\chi$ of size $<\theta_4$, closed under countable sequences, such that $\mathbb{P}_\xi$ and each $\dot{x}_{\alpha,i}$ $(i < \nu_\alpha)$ and $\dot{D}^h_\xi$ $(h \in H)$ are in M. Consider $\mathbb{P}_\xi^- := \mathbb{P}_\xi \cap M$, which is a complete suborder of $\mathbb{P}_\xi$ because the latter has the ccc and M is closed under countable sequences. Then, we force with $\dot{\mathbb{Q}}_\xi := \mathbb{E}^{V^{\mathbb{P}_\xi^-}}$ to advance to the next stage. Note that this is a $\mathbb{P}_\xi^-$-name (for $\mathbb{E}$). Enumerate $\omega^{<\omega} \times \omega = \{(s_k, m_k) \colon k < \omega\}$ and let $\nu_\xi := \omega$ and $\dot{Q}_{\xi,k}$ be a $\mathbb{P}_\xi^-$-name of E_{s_k,m_k} for $k < \omega$.

By the construction of $\mathbb{P}_\xi^-$, for any $h \in H$ we can find a $\mathbb{P}_\xi^-$-name $\dot{D}^{h,-}_\xi$ of $\dot{D}^h_\xi \cap V^{\mathbb{P}_\xi^-}$ (which exists because M is countably closed). Then, Theorem 4.18 applies.

Limit steps are guaranteed by Theorem 4.19, since all the components $Q_{\xi,\zeta}$ are centered. □

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