Blurry definability

Gunter Fuchs
The City University of New York

Set Theory Seminar
KGRC

January 18, 2022

Definition (ZF)

Let $\kappa > 1$ be a cardinal. A set a is *$<\kappa$-blurrily ordinal definable*, or $<\kappa$-OD, if there is an OD set A such that $a \in A$ and $\overline{\overline{A}} < \kappa$. As usual, I will also write $<\kappa$-OD for the class of all sets that are $<\kappa$-OD.

The set a is *hereditarily $<\kappa$-blurrily ordinal definable*, denoted $<\kappa$-HOD, iff $\mathrm{TC}(\{a\}) \subseteq {<\kappa}\text{-OD}$. So $a \in {<\kappa}\text{-HOD}$ iff a is $<\kappa$-OD and $a \subseteq {<\kappa}\text{-HOD}$. Again, I write $<\kappa$-HOD for the class of all $<\kappa$-HOD sets.

Definition (ZF)

Let $\kappa > 1$ be a cardinal. A set a is *$<\kappa$-blurrily ordinal definable*, or $<\kappa$-OD, if there is an OD set A such that $a \in A$ and $\overline{\overline{A}} < \kappa$. As usual, I will also write $<\kappa$-OD for the class of all sets that are $<\kappa$-OD.

The set a is *hereditarily $<\kappa$-blurrily ordinal definable*, denoted $<\kappa$-HOD, iff $\mathrm{TC}(\{a\}) \subseteq {<\kappa}$-OD. So $a \in {<\kappa}$-HOD iff a is $<\kappa$-OD and $a \subseteq {<\kappa}$-HOD. Again, I write $<\kappa$-HOD for the class of all $<\kappa$-HOD sets.

So an object is *$<\kappa$-blurrily ordinal definable* if it is one of fewer than κ objects having a property using ordinal parameters.

History

The case $\kappa = 2$ is (hereditary) ordinal definability (Gödel, Myhill-Scott).

History

The case $\kappa = 2$ is (hereditary) ordinal definability (Gödel, Myhill-Scott).

The case $\kappa = \omega$ is (hereditary) ordinal algebraicity (Hamkins, Leahy).

History

The case $\kappa = 2$ is (hereditary) ordinal definability (Gödel, Myhill-Scott).

The case $\kappa = \omega$ is (hereditary) ordinal algebraicity (Hamkins, Leahy).

The case $\kappa = \omega_1$ was recently proposed and coined (hereditary) "nontypicality" (Tzouvaras).

ZF(C) results

Basics

Proposition

Let $2 \leq \kappa < \lambda$ be cardinals.

1. $\mathrm{OD} \subseteq {<}\kappa\text{-}\mathrm{OD} \subseteq {<}\lambda\text{-}\mathrm{OD}$.
2. ${<}\kappa$-HOD *is transitive, and* $\mathrm{HOD} \subseteq {<}\kappa\text{-}\mathrm{HOD} \subseteq {<}\lambda\text{-}\mathrm{HOD}$.
3. $\mathrm{OD} \cap H_\kappa \subseteq {<}\kappa\text{-}\mathrm{HOD}$.
4. *Under* AC, $H_\kappa \subseteq {<}(2^{<\kappa})^+\text{-}\mathrm{HOD}$.
5. *So under* AC, V *is the increasing union* $\bigcup_{\kappa \in \mathrm{Card}} {<}\kappa\text{-}\mathrm{HOD}$.

Theorem (Hamkins-Leahy)

$<\omega$-HOD $=$ HOA $=$ HOD.

Proposition (ZF)

Let $\kappa \geq 2$ be a cardinal. Then $<\kappa$-HOD is an inner model.

Proof.

Let $\kappa \geq \omega$. It suffices to show that $<\kappa$-HOD satisfies the following condition: for every $u \subseteq <\kappa$-HOD, there is a transitive $v \in <\kappa$-HOD such that $u \subseteq v$ and $\mathrm{Def}(\langle v, \in \rangle) \subseteq <\kappa$-HOD.

Proof.

Let $\kappa \geq \omega$. It suffices to show that $<\kappa$-HOD satisfies the following condition: for every $u \subseteq <\kappa$-HOD, there is a transitive $v \in <\kappa$-HOD such that $u \subseteq v$ and $\mathrm{Def}(\langle v, \in \rangle) \subseteq <\kappa$-HOD.

So let $u \subseteq <\kappa$-HOD be given. Let $u \subseteq V_\alpha$, and set $v = V_\alpha \cap <\kappa$-HOD. Clearly, v is transitive, OD and contained in $<\kappa$-HOD, so $v \in <\kappa$-HOD, and $u \subseteq v$.

Proof.

Let $\kappa \geq \omega$. It suffices to show that $<\kappa$-HOD satisfies the following condition: for every $u \subseteq <\kappa$-HOD, there is a transitive $v \in <\kappa$-HOD such that $u \subseteq v$ and $\mathrm{Def}(\langle v, \in\rangle) \subseteq <\kappa$-HOD.

So let $u \subseteq <\kappa$-HOD be given. Let $u \subseteq V_\alpha$, and set $v = V_\alpha \cap <\kappa$-HOD. Clearly, v is transitive, OD and contained in $<\kappa$-HOD, so $v \in <\kappa$-HOD, and $u \subseteq v$. To show that $\mathrm{Def}(\langle v, \in\rangle) \subseteq <\kappa$-HOD, let $\varphi(x, \vec{y})$ be a formula, and let $\vec{a} = a_0, \ldots, a_{n-1} \in v$. We have to show that $z = \{x \in v \mid \langle v, \in\rangle \models \varphi(x, \vec{a})\} \in <\kappa$-HOD. Since $z \subseteq v \subseteq <\kappa$-HOD, it suffices to show that z is $<\kappa$-OD. For each $i < n$, let A_i be an OD set containing a_i such that $\overline{\overline{A_i}} < \kappa$. Since $a_i \in v$, we may assume that each $b \in A_i$ is in v, by adding this requirement to the definition of A_i if necessary, and this can be done for every $i < n$.

Proof.

Let $\kappa \geq \omega$. It suffices to show that $<\kappa$-HOD satisfies the following condition: for every $u \subseteq <\kappa$-HOD, there is a transitive $v \in <\kappa$-HOD such that $u \subseteq v$ and $\mathrm{Def}(\langle v, \in\rangle) \subseteq <\kappa$-HOD.

So let $u \subseteq <\kappa$-HOD be given. Let $u \subseteq V_\alpha$, and set $v = V_\alpha \cap <\kappa$-HOD. Clearly, v is transitive, OD and contained in $<\kappa$-HOD, so $v \in <\kappa$-HOD, and $u \subseteq v$. To show that $\mathrm{Def}(\langle v, \in\rangle) \subseteq <\kappa$-HOD, let $\varphi(x, \vec{y})$ be a formula, and let $\vec{a} = a_0, \ldots, a_{n-1} \in v$. We have to show that $z = \{x \in v \mid \langle v, \in\rangle \models \varphi(x, \vec{a})\} \in <\kappa$-HOD. Since $z \subseteq v \subseteq <\kappa$-HOD, it suffices to show that z is $<\kappa$-OD. For each $i < n$, let A_i be an OD set containing a_i such that $\overline{\overline{A_i}} < \kappa$. Since $a_i \in v$, we may assume that each $b \in A_i$ is in v, by adding this requirement to the definition of A_i if necessary, and this can be done for every $i < n$. Then z is in the set

$$B = \{w \mid \exists b_0 \in A_0 \ldots \exists b_{n-1} \in A_{n-1} \quad w = \{x \in v \mid \langle v, \in\rangle \models \varphi(x, \vec{b})\}\},$$

B is OD, as $A_0, \ldots, A_{n-1}$ and v are, and obviously, $\overline{\overline{B}} \leq \overline{\overline{A}}_0 \cdot \ldots \cdot \overline{\overline{A}}_{n-1} < \kappa$, as κ is an infinite cardinal. □

Blurry choice

Theorem (ZF)

Let $\kappa > 1$ *be a cardinal. Then, whenever* $C \in {<}\kappa\text{-HOD}$ *is a set consisting of nonempty sets, there is a function* $f : C \longrightarrow ([\bigcup C]^{<\kappa})^{\mathrm{V}}$ *such that* $f \in {<}\kappa\text{-HOD}$*, and such that for every* $c \in C$*,* $\emptyset \neq f(c) \subseteq c$*.*

How close is HOD to blurry HOD?

How close is HOD to blurry HOD?

Two important notions of closeness are Hamkins' approximation and cover properties.

How close is HOD to blurry HOD?

Two important notions of closeness are Hamkins' approximation and cover properties.

Definition

Let $M \subseteq N$ be transitive classes, and let κ be a cardinal in N.

M satisfies the κ-cover property in N if for every set $a \in N$ with $a \subseteq M$ and $\overline{\overline{a}}^N < \kappa$, there is a set $c \in M$ such that $a \subseteq c$ and $\overline{\overline{c}}^M < \kappa$. M satisfies the strong κ-cover property if this is true for every set $a \in N$ with $a \subseteq M$ and $\overline{\overline{a}}^{\mathrm{V}} < \kappa$.

How close is HOD to blurry HOD?

Two important notions of closeness are Hamkins' approximation and cover properties.

Definition

Let $M \subseteq N$ be transitive classes, and let κ be a cardinal in N.

M satisfies the **κ-cover property** in N if for every set $a \in N$ with $a \subseteq M$ and $\overline{\overline{a}}^{N} < \kappa$, there is a set $c \in M$ such that $a \subseteq c$ and $\overline{\overline{c}}^{M} < \kappa$. M satisfies the **strong κ-cover property** if this is true for every set $a \in N$ with $a \subseteq M$ and $\overline{\overline{a}}^{\mathrm{V}} < \kappa$.

Let $a \in N$ be a set with $a \subseteq M$. A set of the form $a \cap c$, where $c \in M$ and $\overline{\overline{c}}^{M} < \kappa$, is called a **$\kappa$-approximation** to a in M. The set a is said to be **κ-approximated in M** if every κ-approximation to a in M belongs to M. M satisfies the **κ-approximation property in N** if whenever $a \in N$ with $a \subseteq M$ is κ-approximated in M, then $a \in M$.

Approximation and cover

Theorem

Let $\kappa \leq \lambda$ be infinite cardinals. Then HOD *satisfies the strong λ-cover property and the λ-approximation property in $<\kappa$-*HOD.

Approximation and cover

Theorem

Let $\kappa \leq \lambda$ be infinite cardinals. Then HOD *satisfies the strong λ-cover property and the λ-approximation property in $<\kappa$-*HOD.

Proof.

For the strong λ-cover property: let $a \in {<\kappa}\text{-HOD}$, $a \subseteq \text{HOD}$, with $\gamma = \overline{\overline{a}} < \lambda$.

Let A be OD with $a \in A$ and $\overline{\overline{A}} < \kappa$. Since $a \subseteq \text{HOD}$ and $\overline{\overline{a}} = \gamma$, we may assume that for all $b \in A$, $b \subseteq \text{HOD}$ and $\overline{\overline{b}} = \gamma$, since these requirements may be added to the definition of A if necessary. Set $c = \bigcup A$. Then $\overline{\overline{c}} \leq \gamma \cdot \overline{\overline{A}} < \lambda$, c is OD, and $c \subseteq \text{HOD}$. Thus, $c \in \text{HOD}$, and clearly, $a \subseteq c$. Since AC holds in HOD, c has a cardinality in HOD, and hence, $\overline{\overline{c}}^{\text{HOD}} < \lambda$, because if it were the case that $\overline{\overline{c}}^{\text{HOD}} \geq \lambda$, then λ would be collapsed as a cardinal. Note that as a consequence, since $\text{HOD} \subseteq {<\kappa}\text{-HOD}$, it is also true in $<\kappa$-HOD that the cardinality of a is less than λ.

Proof (cont'd.)

Concerning the λ-approximation property, since $\lambda \geq \kappa$, it suffices to prove the κ-approximation property.

Proof (cont'd.)

Concerning the λ-approximation property, since $\lambda \geq \kappa$, it suffices to prove the κ-approximation property. So let $a \in {<}\kappa$-HOD, $a \subseteq$ HOD be κ-approximated in HOD. Let A be OD, with $a \in A$ and $\overline{\overline{A}} < \kappa$. We may assume that every $b \in A$ is a subset of HOD that's κ-approximated in HOD.

Proof (cont'd.)

Concerning the λ-approximation property, since $\lambda \geq \kappa$, it suffices to prove the κ-approximation property. So let $a \in {<}\kappa$-HOD, $a \subseteq$ HOD be κ-approximated in HOD. Let A be OD, with $a \in A$ and $\overline{\overline{A}} < \kappa$. We may assume that every $b \in A$ is a subset of HOD that's κ-approximated in HOD. Let $T = \bigcup A$. Then T is OD and $T \subseteq$ HOD, so T is in HOD.

Proof (cont'd.)

Concerning the λ-approximation property, since $\lambda \geq \kappa$, it suffices to prove the κ-approximation property. So let $a \in {<}\kappa\text{-HOD}$, $a \subseteq \text{HOD}$ be κ-approximated in HOD. Let A be OD, with $a \in A$ and $\overline{\overline{A}} < \kappa$. We may assume that every $b \in A$ is a subset of HOD that's κ-approximated in HOD. Let $T = \bigcup A$. Then T is OD and $T \subseteq \text{HOD}$, so T is in HOD.

For every $c \in ([T]^{<\kappa})^{\text{HOD}}$, the set

$$A \sqcap c = \{b \cap c \mid b \in A\}$$

is an OD subset of HOD, and hence an element of HOD.

Proof (cont'd.)

Concerning the λ-approximation property, since $\lambda \geq \kappa$, it suffices to prove the κ-approximation property. So let $a \in {<}\kappa$-HOD, $a \subseteq$ HOD be κ-approximated in HOD. Let A be OD, with $a \in A$ and $\overline{\overline{A}} < \kappa$. We may assume that every $b \in A$ is a subset of HOD that's κ-approximated in HOD. Let $T = \bigcup A$. Then T is OD and $T \subseteq$ HOD, so T is in HOD.

For every $c \in ([T]^{<\kappa})^{\mathrm{HOD}}$, the set

$$A \sqcap c = \{b \cap c \mid b \in A\}$$

is an OD subset of HOD, and hence an element of HOD. Moreover, the function $F : ([T]^{<\kappa})^{\mathrm{HOD}} \longrightarrow$ HOD defined by

$$F(c) = A \sqcap c$$

belongs to HOD as well.

Proof (cont'd).

Define, for distinct $b_0, b_1 \in A$, $d(b_0, b_1)$ to be the least (in the canonical well-ordering of HOD) element of $b_0 \triangle b_1$. Let

$$\Delta = \{d(b_0, b_1) \mid b_0, b_1 \in A,\ b_0 \neq b_1\}.$$

Then $\Delta \in \mathrm{HOD}$, and $\overline{\overline{\Delta}} < \kappa$.

Proof (cont'd).

Define, for distinct $b_0, b_1 \in A$, $d(b_0, b_1)$ to be the least (in the canonical well-ordering of HOD) element of $b_0 \triangle b_1$. Let

$$\Delta = \{d(b_0, b_1) \mid b_0, b_1 \in A,\ b_0 \neq b_1\}.$$

Then $\Delta \in \mathrm{HOD}$, and $\overline{\overline{\Delta}} < \kappa$. Since κ is a cardinal in V, $\overline{\overline{\Delta}}^{\mathrm{HOD}} < \kappa$ as well, so that $\Delta \in ([T]^{<\kappa})^{\mathrm{HOD}}$. Thus, $A \sqcap \Delta \in \mathrm{HOD}$.

Proof (cont'd).

Define, for distinct $b_0, b_1 \in A$, $d(b_0, b_1)$ to be the least (in the canonical well-ordering of HOD) element of $b_0 \triangle b_1$. Let

$$\Delta = \{d(b_0, b_1) \mid b_0, b_1 \in A,\ b_0 \neq b_1\}.$$

Then $\Delta \in \mathrm{HOD}$, and $\overline{\overline{\Delta}} < \kappa$. Since κ is a cardinal in V, $\overline{\overline{\Delta}}^{\mathrm{HOD}} < \kappa$ as well, so that $\Delta \in ([T]^{<\kappa})^{\mathrm{HOD}}$. Thus, $A \sqcap \Delta \in \mathrm{HOD}$. Note that if $b, b' \in A$ are distinct, then $b \cap \Delta \neq b' \cap \Delta$.

Proof (cont'd).

Define, for distinct $b_0, b_1 \in A$, $d(b_0, b_1)$ to be the least (in the canonical well-ordering of HOD) element of $b_0 \triangle b_1$. Let

$$\Delta = \{d(b_0, b_1) \mid b_0, b_1 \in A,\ b_0 \neq b_1\}.$$

Then $\Delta \in \mathrm{HOD}$, and $\overline{\overline{\Delta}} < \kappa$. Since κ is a cardinal in V, $\overline{\overline{\Delta}}^{\mathrm{HOD}} < \kappa$ as well, so that $\Delta \in ([T]^{<\kappa})^{\mathrm{HOD}}$. Thus, $A \sqcap \Delta \in \mathrm{HOD}$. Note that if $b, b' \in A$ are distinct, then $b \cap \Delta \neq b' \cap \Delta$.

As a consequence, for $\Delta \subseteq c \in ([T]^{<\kappa})^{\mathrm{HOD}}$ and $\bar{b} \in A \sqcap \Delta$, there is a unique $\bar{b}' \in A \sqcap c$ such that $\bar{b}' \cap \Delta = \bar{b}$.

Proof (cont'd).

Define, for distinct $b_0, b_1 \in A$, $d(b_0, b_1)$ to be the least (in the canonical well-ordering of HOD) element of $b_0 \triangle b_1$. Let

$$\Delta = \{d(b_0, b_1) \mid b_0, b_1 \in A,\ b_0 \neq b_1\}.$$

Then $\Delta \in \mathrm{HOD}$, and $\overline{\overline{\Delta}} < \kappa$. Since κ is a cardinal in V, $\overline{\overline{\Delta}}^{\mathrm{HOD}} < \kappa$ as well, so that $\Delta \in ([T]^{<\kappa})^{\mathrm{HOD}}$. Thus, $A \sqcap \Delta \in \mathrm{HOD}$. Note that if $b, b' \in A$ are distinct, then $b \cap \Delta \neq b' \cap \Delta$.

As a consequence, for $\Delta \subseteq c \in ([T]^{<\kappa})^{\mathrm{HOD}}$ and $\bar{b} \in A \sqcap \Delta$, there is a unique $\bar{b}' \in A \sqcap c$ such that $\bar{b}' \cap \Delta = \bar{b}$. So we can define in HOD:

$$B(\bar{b}) = \bigcup\{\bar{b}' \mid \exists c \in ([T]^{<\kappa})^{\mathrm{HOD}} \quad (\Delta \subseteq c \text{ and } \bar{b}' \in A \sqcap c \text{ and } \bar{b}' \cap \Delta = \bar{b})\}.$$

Proof (cont'd).

Define, for distinct $b_0, b_1 \in A$, $d(b_0, b_1)$ to be the least (in the canonical well-ordering of HOD) element of $b_0 \triangle b_1$. Let

$$\Delta = \{d(b_0, b_1) \mid b_0, b_1 \in A,\ b_0 \neq b_1\}.$$

Then $\Delta \in \mathrm{HOD}$, and $\overline{\overline{\Delta}} < \kappa$. Since κ is a cardinal in V, $\overline{\overline{\Delta}}^{\mathrm{HOD}} < \kappa$ as well, so that $\Delta \in ([T]^{<\kappa})^{\mathrm{HOD}}$. Thus, $A \sqcap \Delta \in \mathrm{HOD}$. Note that if $b, b' \in A$ are distinct, then $b \cap \Delta \neq b' \cap \Delta$.

As a consequence, for $\Delta \subseteq c \in ([T]^{<\kappa})^{\mathrm{HOD}}$ and $\bar{b} \in A \sqcap \Delta$, there is a unique $\bar{b}' \in A \sqcap c$ such that $\bar{b}' \cap \Delta = \bar{b}$. So we can define in HOD:

$$B(\bar{b}) = \bigcup \{\bar{b}' \mid \exists c \in ([T]^{<\kappa})^{\mathrm{HOD}} \quad (\Delta \subseteq c \text{ and } \bar{b}' \in A \sqcap c \text{ and } \bar{b}' \cap \Delta = \bar{b})\}.$$

It follows that $B(\bar{b})$ is the unique $b \in A$ such that $b \cap \Delta = \bar{b}$. Since for $b \in A$, $b = B(b \cap \Delta)$, it follows that

$$A = \{B(\bar{b}) \mid \bar{b} \in A \sqcap \Delta\}$$

and hence, $A \in \mathrm{HOD}$. In particular, $a \in \mathrm{HOD}$. □

Consequences

Note that the fact that HOD satisfies the ω-approximation property in $<\omega$-HOD immediately implies the Hamkins-Leahy result that HOD $= <\omega$-HOD.

Consequences

Note that the fact that HOD satisfies the ω-approximation property in $<\omega$-HOD immediately implies the Hamkins-Leahy result that HOD $= <\omega$-HOD.

Proposition

Let κ be an infinite cardinal. If θ is a limit ordinal with $\mathrm{cf}^{<\kappa\text{-HOD}}(\theta) \geq \kappa$, then $<\kappa$-HOD has no length θ sequence that's fresh over HOD.

Bukovský's condition

Definition (Bukovský)

Let $M_1 \subseteq M_2$ be transitive models, and let κ be a cardinal in M_2. Then $\mathrm{Apr}_{M_1,M_2}(\kappa)$ says that whenever $f \in M_2$ is a function from an ordinal α to an ordinal β, then there is a function $g : \alpha \longrightarrow \mathcal{P}(\beta)$ in M_1 such that for every $\xi < \alpha$, $f(\xi) \in g(\xi)$ and $\overline{\overline{g(\xi)}}^{M_1} < \kappa$.

Bukovský's condition

Definition (Bukovský)

Let $M_1 \subseteq M_2$ be transitive models, and let κ be a cardinal in M_2. Then $\mathrm{Apr}_{M_1,M_2}(\kappa)$ says that whenever $f \in M_2$ is a function from an ordinal α to an ordinal β, then there is a function $g : \alpha \longrightarrow \mathcal{P}(\beta)$ in M_1 such that for every $\xi < \alpha$, $f(\xi) \in g(\xi)$ and $\overline{\overline{g(\xi)}}^{M_1} < \kappa$.

The remarkable main theorem of Bukovský on this condition is the following.

Bukovský's condition

Definition (Bukovský)

Let $M_1 \subseteq M_2$ be transitive models, and let κ be a cardinal in M_2. Then $\mathrm{Apr}_{M_1,M_2}(\kappa)$ says that whenever $f \in M_2$ is a function from an ordinal α to an ordinal β, then there is a function $g : \alpha \longrightarrow \mathcal{P}(\beta)$ in M_1 such that for every $\xi < \alpha$, $f(\xi) \in g(\xi)$ and $\overline{\overline{g(\xi)}}^{M_1} < \kappa$.

The remarkable main theorem of Bukovský on this condition is the following.

Theorem (ZFC, Bukovský 1973)

Suppose M is a transitive inner model of ZFC, and κ is an infinite cardinal. Then the following conditions are equivalent:

1. V *is a forcing extension of M by a κ-c.c. forcing notion.*
2. $\mathrm{Apr}_{M,\mathrm{V}}(\kappa)$ *holds.*

Proposition

Let κ be a cardinal. Then $\mathrm{Apr}_{\mathrm{HOD},<\kappa\text{-}\mathrm{HOD}}(\kappa)$ holds.

Proposition

Let κ be a cardinal. Then $\mathrm{Apr}_{\mathrm{HOD},<\kappa\text{-HOD}}(\kappa)$ *holds.*

Proof.

I will prove more: if $f : d \longrightarrow \mathrm{HOD}$ is a function in $<\kappa$-HOD with $d \in \mathrm{HOD}$, then there is in HOD a function $g : d \longrightarrow \mathrm{HOD}$ such that for every $x \in d$, $f(x) \in g(x)$ and $\overline{\overline{g(x)}}^{\mathrm{HOD}} < \kappa$.

Proposition

Let κ be a cardinal. Then $\mathrm{Apr}_{\mathrm{HOD},<\kappa\text{-HOD}}(\kappa)$ *holds.*

Proof.

I will prove more: if $f : d \longrightarrow \mathrm{HOD}$ is a function in $<\kappa$-HOD with $d \in \mathrm{HOD}$, then there is in HOD a function $g : d \longrightarrow \mathrm{HOD}$ such that for every $x \in d$, $f(x) \in g(x)$ and $\overline{\overline{g(x)}}^{\mathrm{HOD}} < \kappa$.

To see this, let f be as described. Let F be OD with $f \in F$ and $\overline{\overline{F}}^{\mathrm{V}} < \kappa$, and such that for every $g \in F$, $g : d \longrightarrow \mathrm{HOD}$.

Proposition

Let κ be a cardinal. Then $\mathrm{Apr}_{\mathrm{HOD},<\kappa\text{-}\mathrm{HOD}}(\kappa)$ *holds.*

Proof.

I will prove more: if $f : d \longrightarrow \mathrm{HOD}$ is a function in $<\kappa$-HOD with $d \in \mathrm{HOD}$, then there is in HOD a function $g : d \longrightarrow \mathrm{HOD}$ such that for every $x \in d$, $f(x) \in g(x)$ and $\overline{\overline{g(x)}}^{\mathrm{HOD}} < \kappa$.

To see this, let f be as described. Let F be OD with $f \in F$ and $\overline{\overline{F}}^{\mathrm{V}} < \kappa$, and such that for every $g \in F$, $g : d \longrightarrow \mathrm{HOD}$. Define a function g with domain d by

$$g(x) = \{h(x) \mid h \in F\}.$$

Proposition

Let κ be a cardinal. Then $\mathrm{Apr}_{\mathrm{HOD},<\kappa\text{-HOD}}(\kappa)$ *holds.*

Proof.

I will prove more: if $f : d \longrightarrow \mathrm{HOD}$ is a function in $<\kappa$-HOD with $d \in \mathrm{HOD}$, then there is in HOD a function $g : d \longrightarrow \mathrm{HOD}$ such that for every $x \in d$, $f(x) \in g(x)$ and $\overline{\overline{g(x)}}^{\mathrm{HOD}} < \kappa$.

To see this, let f be as described. Let F be OD with $f \in F$ and $\overline{\overline{F}}^{\mathrm{V}} < \kappa$, and such that for every $g \in F$, $g : d \longrightarrow \mathrm{HOD}$. Define a function g with domain d by

$$g(x) = \{h(x) \mid h \in F\}.$$

Then $g \in \mathrm{HOD}$, and for $x \in d$, $f(x) \in g(x)$ and $\overline{\overline{g(x)}}^{\mathrm{HOD}} < \kappa$. So g is as wished. □

Consequences

Theorem

Let κ *be an infinite cardinal. Then the following are equivalent:*

1. $<\kappa$-HOD *satisfies the axiom of choice.*

Consequences

Theorem

Let κ be an infinite cardinal. Then the following are equivalent:

1. $<\kappa$-HOD *satisfies the axiom of choice.*
2. $<\kappa$-HOD *is a set forcing extension of* HOD *by a* κ*-c.c. forcing notion.*

Consequences

Theorem

Let κ be an infinite cardinal. Then the following are equivalent:

1. $<\kappa$-HOD *satisfies the axiom of choice.*
2. $<\kappa$-HOD *is a set forcing extension of* HOD *by a* κ*-c.c. forcing notion.*

Proposition

Let κ be an infinite cardinal. Then HOD *and* $<\kappa$-HOD *have the same cardinals and cofinalities above* κ*, in the following sense:*

1. *If λ is a limit ordinal such that* $\mathrm{cf}^{\mathsf{HOD}}(\lambda) \geq \kappa$*, then* $\mathrm{cf}^{\mathsf{HOD}}(\lambda) = \mathrm{cf}^{<\kappa\text{-}\mathsf{HOD}}(\lambda)$.

Consequences

Theorem

Let κ be an infinite cardinal. Then the following are equivalent:

1. $<\kappa$-HOD *satisfies the axiom of choice.*
2. $<\kappa$-HOD *is a set forcing extension of* HOD *by a* κ*-c.c. forcing notion.*

Proposition

Let κ be an infinite cardinal. Then HOD *and* $<\kappa$-HOD *have the same cardinals and cofinalities above* κ*, in the following sense:*

1. *If λ is a limit ordinal such that* $\mathrm{cf}^{\mathsf{HOD}}(\lambda) \geq \kappa$*, then* $\mathrm{cf}^{\mathsf{HOD}}(\lambda) = \mathrm{cf}^{<\kappa\text{-}\mathsf{HOD}}(\lambda)$.
2. *For $\lambda \geq \kappa$, λ is regular in* HOD *iff λ is regular in* $<\kappa$-HOD.

Consequences

Theorem

Let κ be an infinite cardinal. Then the following are equivalent:

1. $<\kappa$-HOD *satisfies the axiom of choice.*
2. $<\kappa$-HOD *is a set forcing extension of* HOD *by a* κ*-c.c. forcing notion.*

Proposition

Let κ be an infinite cardinal. Then HOD *and* $<\kappa$-HOD *have the same cardinals and cofinalities above* κ*, in the following sense:*

1. *If λ is a limit ordinal such that* $\mathrm{cf}^{\mathsf{HOD}}(\lambda) \geq \kappa$*, then* $\mathrm{cf}^{\mathsf{HOD}}(\lambda) = \mathrm{cf}^{<\kappa\text{-}\mathsf{HOD}}(\lambda)$.
2. *For* $\lambda \geq \kappa$*,* λ *is regular in* HOD *iff* λ *is regular in* $<\kappa$-HOD.
3. $\mathrm{Card}^{\mathsf{HOD}} \setminus \kappa = \mathrm{Card}^{<\kappa\text{-}\mathsf{HOD}} \setminus \kappa$.

Hamkins-Laver without choice

Theorem

Let W be a transitive model satisfying ZF*. Let κ be a regular cardinal in W. Let $\mathcal{M}, \mathcal{M}' \in W$ be transitive models of* ZF *without replacement, with $\theta = \mathcal{M} \cap \mathrm{On} = \mathcal{M}' \cap \mathrm{On}$, and each satisfying: if $\bar{\kappa} \leq \kappa$ and $r \subseteq \bar{\kappa} \times \bar{\kappa}$ is such that $\langle \bar{\kappa}, r \rangle$ a well-order, then there are an ordinal α and a function $\pi : \bar{\kappa} \longrightarrow \alpha$ such that $\pi : \langle \bar{\kappa}, r \rangle \longrightarrow \langle \alpha, < \rangle$ is an isomorphism, and every set of ordinals has a monotone enumeration. Moreover, suppose both $\mathcal{M}$ and $\mathcal{M}'$ satisfy following condition, a form of the axiom of choice: if $\mathcal{X} = \mathcal{M}$ or $\mathcal{M}'$ and $a \in \mathcal{X}$, then there is an $r \subseteq a \times a$ in $\mathcal{X}$ such that in W (equivalently, in V), $\langle a, r \rangle$ is a well-order.*

Let $W_\theta = (\mathrm{V}_\theta)^W$, and suppose that both $\mathcal{M}$ and $\mathcal{M}'$ satisfy the κ-cover and approximation properties in W_θ. Suppose, moreover, that $\mathcal{P}(\kappa) \cap \mathcal{M} = \mathcal{P}(\kappa) \cap \mathcal{M}'$, and that $(\kappa^+)^{\mathcal{M}} = (\kappa^+)^{\mathcal{M}'} = (\kappa^+)^W$.

Then it follows that $\mathcal{P}(\theta) \cap \mathcal{M} = \mathcal{P}(\theta) \cap \mathcal{M}'$.

More consequences

Theorem

Let $\lambda \geq 2$ be a cardinal. Let $\kappa \geq \lambda$ be regular. Then HOD *is definable in* $<\lambda$-HOD *using* $\mathcal{P}(\kappa) \cap$ HOD *as a parameter.*

More consequences

Theorem

Let $\lambda \geq 2$ be a cardinal. Let $\kappa \geq \lambda$ be regular. Then HOD *is definable in* $<\lambda$-HOD *using* $\mathcal{P}(\kappa) \cap \text{HOD}$ *as a parameter.*

Proposition

Let κ be an infinite cardinal, and let $\lambda \geq \kappa$ be inaccessible in HOD.

1. *If λ weakly compact in* $<\kappa$-HOD*, then it is weakly compact in* HOD.
2. *If λ is measurable in* $<\kappa$-HOD*, then it is measurable in* HOD.

Capturing large cardinals

This shows that $<\kappa$-HOD does not capture that much more of the large cardinal structure of V than HOD does. On the one hand, if κ is inaccessible, then $\mathrm{V}_\kappa \cap <\kappa\text{-HOD} = \mathrm{V}_\kappa$, so that large cardinal properties of V witnessed by V_κ are inherited by $<\kappa$-HOD. But it was shown by Cheng, Friedman and Hamkins that it is consistent that a supercompact cardinal λ is not weakly compact in HOD- so if $\kappa \leq \lambda$, then λ is not weakly compact in $<\kappa$-HOD either.

Leaps

Definition

A cardinal $\lambda > 2$ is a leap if

$$<\delta\text{-HOD} \subsetneq <\lambda\text{-HOD},$$

for every cardinal $\delta < \lambda$. I write $\langle \Lambda_\alpha \mid \alpha < \Theta \rangle$ for the monotone enumeration of the leaps.

Leaps

Definition

A cardinal $\lambda > 2$ is a leap if

$$<\delta\text{-HOD} \subsetneq <\lambda\text{-HOD},$$

for every cardinal $\delta < \lambda$. I write $\langle \Lambda_\alpha \mid \alpha < \Theta \rangle$ for the monotone enumeration of the leaps.

Lemma

Leaps have the following properties.

1. *The class of leaps is closed in the ordinals.*
2. Λ_0*, if defined, is an uncountable successor cardinal.*
3. *Successor leaps are successor cardinals.*

Big leaps

Definition

Say that a leap γ is a big leap if

$$\left(\bigcup_{\delta<\gamma,\delta\in\mathrm{Card}} {<}\delta\text{-HOD}\right) \subsetneq {<}\gamma\text{-HOD}.$$

Big leaps

Definition

Say that a leap γ is a big leap if

$$\left(\bigcup_{\delta<\gamma,\delta\in\mathrm{Card}} {<}\delta\text{-HOD}\right) \subsetneq {<}\gamma\text{-HOD}.$$

Theorem

Every leap is big.

Big leaps

Definition

Say that a leap γ is a big leap if

$$\left(\bigcup_{\delta<\gamma,\delta\in\mathrm{Card}} {<}\delta\text{-HOD}\right) \subsetneq {<}\gamma\text{-HOD}.$$

Theorem

Every leap is big.

More info on limit leaps:

Theorem

If λ is a limit leap, then ${<}\lambda$-HOD does not satisfy the axiom of choice.

Proof.

Let

$$T = \{\kappa < \lambda \mid \kappa \text{ is a successor leap}\}.$$

For a successor leap κ, let κ_- be its predecessor leap. For $\kappa \in T$, let $\tau_\kappa = \langle \ulcorner \varphi_\kappa \urcorner, \alpha_\kappa, \beta_\kappa \rangle$ be the least code for an $\in$-minimal element of $<\kappa$-HOD $\setminus$ $<\kappa_-$-HOD.

Proof.

Let

$$T = \{\kappa < \lambda \mid \kappa \text{ is a successor leap}\}.$$

For a successor leap κ, let κ_- be its predecessor leap. For $\kappa \in T$, let $\tau_\kappa = \langle \ulcorner\varphi_\kappa\urcorner, \alpha_\kappa, \beta_\kappa\rangle$ be the least code for an $\in$-minimal element of $<\kappa$-HOD $\setminus$ $<\kappa_-$-HOD. That is, letting $A_\kappa = \{x \mid \mathrm{Sat}(\mathrm{V}_{\alpha_\kappa}, \ulcorner\varphi_\kappa\urcorner, \beta_\kappa)\}$, A_κ has cardinality less than κ, there is an $a \in A_\kappa$ such that a is $\in$-minimal in $<\kappa$-HOD $\setminus$ $<\kappa_-$-HOD, and τ_κ is minimal with these properties.

Proof.

Let

$$T = \{\kappa < \lambda \mid \kappa \text{ is a successor leap}\}.$$

For a successor leap κ, let κ_- be its predecessor leap. For $\kappa \in T$, let $\tau_\kappa = \langle \ulcorner\varphi_\kappa\urcorner, \alpha_\kappa, \beta_\kappa\rangle$ be the least code for an $\in$-minimal element of $<\kappa$-HOD $\setminus$ $<\kappa_-$-HOD. That is, letting $A_\kappa = \{x \mid \mathrm{Sat}(\mathrm{V}_{\alpha_\kappa}, \ulcorner\varphi_\kappa\urcorner, \beta_\kappa)\}$, A_κ has cardinality less than κ, there is an $a \in A_\kappa$ such that a is $\in$-minimal in $<\kappa$-HOD $\setminus$ $<\kappa_-$-HOD, and τ_κ is minimal with these properties. Let

$$B_\kappa = \{x \in A_\kappa \mid x \text{ is } \in\text{-minimal in } {<}\kappa\text{-HOD} \setminus {<}\kappa_-\text{-HOD}\}.$$

Proof.

Let

$$T = \{\kappa < \lambda \mid \kappa \text{ is a successor leap}\}.$$

For a successor leap κ, let κ_- be its predecessor leap. For $\kappa \in T$, let $\tau_\kappa = \langle \ulcorner \varphi_\kappa \urcorner, \alpha_\kappa, \beta_\kappa \rangle$ be the least code for an $\in$-minimal element of $<\kappa$-HOD $\setminus$ $<\kappa_-$-HOD. That is, letting $A_\kappa = \{x \mid \mathrm{Sat}(\mathrm{V}_{\alpha_\kappa}, \ulcorner \varphi_\kappa \urcorner, \beta_\kappa)\}$, A_κ has cardinality less than κ, there is an $a \in A_\kappa$ such that a is $\in$-minimal in $<\kappa$-HOD $\setminus$ $<\kappa_-$-HOD, and τ_κ is minimal with these properties. Let

$$B_\kappa = \{x \in A_\kappa \mid x \text{ is } \in\text{-minimal in } {<}\kappa\text{-HOD} \setminus {<}\kappa_-\text{-HOD}\}.$$

So for every $b \in B_\kappa$, $b \in {<}\kappa$-HOD, $b \notin {<}\kappa_-$-HOD, but $b \subseteq {<}\kappa_-$-HOD.

Proof.

Let

$$T = \{\kappa < \lambda \mid \kappa \text{ is a successor leap}\}.$$

For a successor leap κ, let κ_- be its predecessor leap. For $\kappa \in T$, let $\tau_\kappa = \langle \ulcorner \varphi_\kappa \urcorner, \alpha_\kappa, \beta_\kappa \rangle$ be the least code for an $\in$-minimal element of $<\kappa$-HOD $\setminus$ $<\kappa_-$-HOD. That is, letting $A_\kappa = \{x \mid \mathrm{Sat}(\mathrm{V}_{\alpha_\kappa}, \ulcorner \varphi_\kappa \urcorner, \beta_\kappa)\}$, A_κ has cardinality less than κ, there is an $a \in A_\kappa$ such that a is $\in$-minimal in $<\kappa$-HOD $\setminus$ $<\kappa_-$-HOD, and τ_κ is minimal with these properties. Let

$$B_\kappa = \{x \in A_\kappa \mid x \text{ is } \in\text{-minimal in } {<}\kappa\text{-HOD} \setminus {<}\kappa_-\text{-HOD}\}.$$

So for every $b \in B_\kappa$, $b \in$ $<\kappa$-HOD, $b \notin$ $<\kappa_-$-HOD, but $b \subseteq$ $<\kappa_-$-HOD. $\vec{B} = \langle B_\kappa \mid \kappa \in T \rangle$ is OD, and $\vec{B}$ belongs to $<\lambda$-HOD, but it is not in $<\bar{\kappa}$-HOD for any cardinal $\bar{\lambda} < \lambda$ (showing that λ is a big leap).

Proof (cont'd).

$\vec{B}$ is a sequence of nonempty sets, and I claim that it has no choice function in $<\lambda$-HOD: suppose it did. Let $\vec{b} = \langle b_\kappa \mid \kappa \in T \rangle \in {<\lambda}$-HOD be such that for every $\kappa \in T$, $b_\kappa \in B_\kappa$. Since $\vec{b} \in {<\lambda}$-HOD, it is $<\gamma$-OD, for some cardinal $\gamma < \lambda$. Let X witness this, that is let X be OD and of cardinality less than γ, so that $\vec{b} \in X$. Let

$$Y = X \cap \prod_{\kappa \in T} B_\kappa.$$

Clearly, Y is still OD, has cardinality less than γ, and has $\vec{b} \in Y$.

Proof (cont'd).

$\vec{B}$ is a sequence of nonempty sets, and I claim that it has no choice function in $<\lambda$-HOD: suppose it did. Let $\vec{b} = \langle b_\kappa \mid \kappa \in T \rangle \in {<}\lambda$-HOD be such that for every $\kappa \in T$, $b_\kappa \in B_\kappa$. Since $\vec{b} \in {<}\lambda$-HOD, it is $<\gamma$-OD, for some cardinal $\gamma < \lambda$. Let X witness this, that is let X be OD and of cardinality less than γ, so that $\vec{b} \in X$. Let

$$Y = X \cap \prod_{\kappa \in T} B_\kappa.$$

Clearly, Y is still OD, has cardinality less than γ, and has $\vec{b} \in Y$.

Now pick $\kappa \in T$ such that $\gamma \leq \kappa_-$. I claim that $b_\kappa \in {<}\kappa_-$-HOD, a contradiction. Namely, let

$$Z = \{x(\kappa) \mid x \in Y\}.$$

This is an OD set of cardinality at most $\overline{\overline{Y}} < \gamma \leq \kappa_-$, and b_κ belongs to it. So b_κ is $<\kappa_-$-OD. And since $b_\kappa \in B_\kappa$, we know that $b_\kappa \subseteq {<}\kappa_-$-HOD. Thus, $b_\kappa \in {<}\kappa_-$-HOD, as claimed. □

Consistency results

Preserving membership to blurry HOD

Proposition (ZFC)

Suppose that $\mathbb{P}$ *is a notion of forcing,* G *is generic for* $\mathbb{P}$ *over* V, κ *is a cardinal in* $\mathrm{V}[G]$, *and* V *is definable in* $\mathrm{V}[G]$ *from a parameter in* ${<}\kappa\text{-OD}^{\mathrm{V}[G]}$. *Then*

$$<\kappa\text{-OD}^{\mathrm{V}} \subseteq <\kappa\text{-OD}^{\mathrm{V}[G]}$$

and so, $<\kappa\text{-HOD}^{\mathrm{V}} \subseteq <\kappa\text{-HOD}^{\mathrm{V}[G]}$ *as well.*

Preserving membership to blurry HOD

Proposition (ZFC)

Suppose that $\mathbb{P}$ is a notion of forcing, G is generic for $\mathbb{P}$ over V, κ is a cardinal in $\mathrm{V}[G]$, and V is definable in $\mathrm{V}[G]$ from a parameter in $<\kappa\text{-OD}^{\mathrm{V}[G]}$. Then

$$<\kappa\text{-OD}^{\mathrm{V}} \subseteq <\kappa\text{-OD}^{\mathrm{V}[G]}$$

and so, $<\kappa\text{-HOD}^{\mathrm{V}} \subseteq <\kappa\text{-HOD}^{\mathrm{V}[G]}$ as well.

Corollary

Let κ be a cardinal, and let $\mathbb{P}$ be a notion of forcing of cardinality γ, where $2^{(2^\gamma)} < \kappa$. If G is $\mathbb{P}$-generic over V, then

$$<\kappa\text{-HOD}^{\mathrm{V}} \subseteq <\kappa\text{-HOD}^{\mathrm{V}[G]}.$$

Homogeneity

Definition (à la Dobrinen-Friedman)

Let $\mathbb{P}$ be a forcing notion. For $p \in \mathbb{P}$, let the cone below p in $\mathbb{P}$ be the set

$$\mathbb{P}_{\leq p} = \{q \in \mathbb{P} \mid q \leq p\}$$

equipped with the restriction of the ordering of $\mathbb{P}$.

$\mathbb{P}$ is called cone homogeneous if for any two conditions $p, q \in \mathbb{P}$, there are $p' \leq p$ and $q' \leq q$ such that $\mathbb{P}_{\leq p'}$ and $\mathbb{P}_{\leq q'}$ are isomorphic.

Lemma

Let κ be a regular cardinal, $\mathbb{P}$ a cone homogeneous, $<\kappa$-closed forcing notion, and let $G \subseteq \mathbb{P}$ be $\mathbb{P}$-generic over V. Then

$$<\kappa\text{-}\mathsf{HOD}^{\mathrm{V}[G]} \subseteq \mathrm{V}.$$

Lemma

Let κ be a regular cardinal, $\mathbb{P}$ a cone homogeneous, $<\kappa$-closed forcing notion, and let $G \subseteq \mathbb{P}$ be $\mathbb{P}$-generic over V. Then

$$<\kappa\text{-HOD}^{\mathrm{V}[G]} \subseteq \mathrm{V}.$$

Note how nicely this lemma generalizes the folklore fact that if $\mathbb{P}$ is cone homogeneous and $G \subseteq \mathbb{P}$ is generic, then $\mathrm{HOD}^{\mathrm{V}[G]} \subseteq \mathrm{V}$ - this is the special case $\kappa = \omega$.

Not adding to blurry HOD

Lemma

Let κ be a regular cardinal, $\mathbb{P}$ a cone homogeneous, $<\kappa$-closed forcing notion, $\bar{\kappa} \leq \kappa$ a cardinal such that $\mathbb{P}$ is $<\bar{\kappa}$-OD, and let $G \subseteq \mathbb{P}$ be $\mathbb{P}$-generic over V. Then

$$<\bar{\kappa}\text{-HOD}^{\mathrm{V}[G]} \subseteq <\bar{\kappa}\text{-HOD}^{\mathrm{V}}.$$

Not adding to blurry HOD

Lemma

Let κ be a regular cardinal, $\mathbb{P}$ a cone homogeneous, $<\kappa$-closed forcing notion, $\bar{\kappa} \leq \kappa$ a cardinal such that $\mathbb{P}$ is $<\bar{\kappa}$-OD, and let $G \subseteq \mathbb{P}$ be $\mathbb{P}$-generic over V. Then

$$<\bar{\kappa}\text{-HOD}^{\mathrm{V}[G]} \subseteq <\bar{\kappa}\text{-HOD}^{\mathrm{V}}.$$

Again, note how nicely this generalizes the fact that if $\mathbb{P}$ is and OD forcing notion and $G \subseteq \mathbb{P}$ is generic, then $\mathrm{HOD}^{\mathrm{V}[G]} \subseteq \mathrm{HOD}^{\mathrm{V}}$.

Cohen forcing

Theorem (ZFC)

Let κ be an infinite regular cardinal such that $\kappa^{<\kappa} = \kappa$, and let G be generic for $\mathbb{P} = \mathrm{Add}(\kappa, 1)$. If $\bar{\kappa}$ is a cardinal less than or equal to 2^κ in $\mathrm{V}[G]$, then

$$<\bar{\kappa}\text{-HOD}^{\mathrm{V}[G]} \subseteq <\bar{\kappa}\text{-HOD}^{\mathrm{V}}.$$

Corollary

Assume $\mathrm{V} = L$*, and let* κ *be an infinite regular cardinal. If* G *is generic for* $\mathbb{P} = \mathrm{Add}(\kappa, 1)$*, then*

$$L = \mathrm{HOD}^{L[G]} = {<}\kappa^{+}\text{-}\mathrm{HOD}^{L[G]} \subsetneq {<}\kappa^{++}\text{-}\mathrm{HOD}^{L[G]} = L[G].$$

In particular, $\Lambda_0^{L[G]} = \kappa^{++}$.

Iterated Cohen forcing

Theorem

Assume $\mathrm{V} = L$*. Let* λ *be a cardinal, and let* $\langle\langle \mathbb{P}_i \mid i \leq \lambda\rangle, \langle \dot{\mathbb{Q}}_i \mid i < \lambda\rangle\rangle$ *be the reverse Easton iteration whose only nontrivial stages are when* $i = \kappa$ *is an infinite regular cardinal,* $\Vdash_{\mathbb{P}_\kappa} \dot{\mathbb{Q}}_\kappa = \mathrm{Add}(\kappa, 1)$*. Let* G *be* $\mathbb{P} = \mathbb{P}_\lambda$*-generic over* L*. Then:*

1. *for regular* $\kappa < \lambda$, $L[G{\restriction}(\kappa+1)] = {<}\kappa^{++}\text{-HOD}^{L[G]}$*, and* $G(\kappa) \in {<}\kappa^{++}\text{-HOD}^{L[G]} \setminus {<}\kappa^{+}\text{-HOD}^{L[G]}$.
2. $L = {<}\omega_1\text{-HOD}^{L[G]}$*, so* ω_1 *is not a leap in* $L[G]$.
3. *for any limit cardinal* $\kappa \leq \lambda$, $G{\restriction}\kappa \in {<}\kappa^{++}\text{-HOD}^{L[G]} \setminus {<}\kappa^{+}\text{-HOD}^{L[G]}$. *So* κ^{++} *is a leap in* $L[G]$.

The forcing of Kanovei-Lyubetsky

Kanovei and Lyubetsky formed the finite support product $\mathbb{T}^{<\omega}$ of a forcing in L, due to Jensen, whose conditions are a certain collection of perfect trees, ordered by inclusion. This product is a ccc forcing in L, and forcing with $\mathbb{T}^{<\omega}$ over L adds a sequence $\vec{x} = \langle x_i \mid i < \omega \rangle$ of reals such that in $L[\vec{x}]$, $\{x_i \mid i < \omega\}$ is the set of $\mathbb{T}$-generic reals over L.

Proposition

Let $\vec{x}$ be a $\mathbb{T}^{<\omega}$-generic sequence over L. Then

1. $\{x_i \mid i < \omega\}$ *is* $\mathrm{OD}^{L[\vec{x}]}$,
2. $\{x_i \mid i < \omega\} \subseteq {<}\omega_1\text{-HOD}^{L[\vec{x}]}$,
3. $\{x_i \mid i < \omega\} \in {<}\omega_1\text{-HOD}^{L[\vec{x}]}$,
4. $\vec{x} \notin {<}\omega_1\text{-HOD}^{L[\vec{x}]}$,
5. *in* $L[\vec{x}]$, $L = \mathrm{HOD} = {<}\omega\text{-HOD} \subsetneq {<}\omega_1\text{-HOD} \subsetneq {<}\omega_2\text{-HOD} = \mathrm{V}$. *In particular,* $\Lambda_0^{L[\vec{x}]} = \omega_1$.

Iterating

Corollary

Assume $\mathrm{V} = L$. Let $\vec{x}$ be a $\mathbb{T}^{<\omega}$-generic sequence over L. Working in $L[\vec{x}]$, let λ be a cardinal, and let $\langle\langle \mathbb{P}_i \mid i \leq \lambda\rangle, \langle \dot{\mathbb{Q}}_i \mid i < \lambda\rangle\rangle$ be the reverse Easton iteration whose only nontrivial stages are when $i = \kappa$ is an uncountable regular cardinal, in which case $\Vdash_{\mathbb{P}_\kappa} \dot{\mathbb{Q}}_\kappa = \mathrm{Add}(\kappa, 1)$. Let G be $\mathbb{P} = \mathbb{P}_\lambda$-generic over $L[\vec{x}]$. Then:

1. *for uncountable regular $\kappa < \lambda$, $L[\vec{x}][G{\restriction}(\kappa + 1)] = {<}\kappa^{++}\text{-HOD}^{L[\vec{x}][G]}$, and $G(\kappa) \in {<}\kappa^{++}\text{-HOD}^{L[\vec{x}][G]} \setminus {<}\kappa^{+}\text{-HOD}^{L[\vec{x}][G]}$.*
2. $L[\vec{x}] = {<}\omega_2\text{-HOD}^{L[\vec{x}][G]}$
3. *for any limit cardinal $\kappa \leq \lambda$, $G{\restriction}\kappa \in {<}\kappa^{++}\text{-HOD}^{L[\vec{x}][G]} \setminus {<}\kappa^{+}\text{-HOD}^{L[\vec{x}][G]}$.*
4. $\{x_i \mid i < \omega\} \in {<}\omega_1\text{-HOD}^{L[\vec{x}][G]}$, *but* $\vec{x} \notin {<}\omega_1\text{-HOD}^{L[\vec{x}][G]}$.
5. ${<}\omega\text{-HOD}^{L[\vec{x}][G]} = L$.

Remark

Thus, in $L[\vec{x}][G]$, if $\omega_1 \leq \kappa \leq \lambda$, and either κ is regular and $\kappa < \lambda$, or κ is a limit cardinal, then κ^{++} is a leap. All limit cardinals up to λ are also leaps, and so are ω_1 and ω_2.

For example, if $\lambda = \aleph_\omega$, then in $L[\vec{x}][G]$, all uncountable cardinals up to and including $\aleph_\omega$ are leaps, as is $\aleph_{\omega+2}$.

Remark

Thus, in $L[\vec{x}][G]$, if $\omega_1 \leq \kappa \leq \lambda$, and either κ is regular and $\kappa < \lambda$, or κ is a limit cardinal, then κ^{++} is a leap. All limit cardinals up to λ are also leaps, and so are ω_1 and ω_2.

For example, if $\lambda = \aleph_\omega$, then in $L[\vec{x}][G]$, all uncountable cardinals up to and including $\aleph_\omega$ are leaps, as is $\aleph_{\omega+2}$.

The natural question is how to arrange the successor cardinal of a limit leap to be a leap, or even how to arrange that the least leap is the successor of a limit cardinal (it can't be a limit cardinal).

Homogeneous Souslin trees

Definition (á la Brodksy-Rinot)

Let κ be a regular cardinal. A streamlined (or sequential) κ-tree is a set T of functions p such that the domain of p is an ordinal less than κ and the range of p is contained in κ, closed under restrictions to ordinals, ordered by inclusion, such that for every $\alpha < \kappa$, the α-the level of T, $T(\alpha) = \{p \in T \mid \mathrm{dom}(p) = \alpha\}$ has cardinality less than κ and is nonempty. If $p, q \in T$, then $p \perp q$ (p, q are incompatible) iff neither $p \subseteq q$ nor $q \subseteq p$. An antichain in T is a set $A \subseteq T$ of pairwise incompatible elements. T is a κ-Souslin tree if it has no antichain of cardinality κ. It is coherent if whenever $p, q \in T$, then the set $d(p, q) = \{i \in \mathrm{dom}(p) \cap \mathrm{dom}(q) \mid p(i) \neq q(i)\}$ is finite. It is uniformly homogeneous if whenever $p, q \in T$ and $\mathrm{dom}(p) \leq \mathrm{dom}(q)$, then the function $p * q = p \cup (q \restriction (\mathrm{dom}(q) \setminus \mathrm{dom}(p))) \in T$. It is uniformly coherent if it is coherent and uniformly homogeneous.

Creating a leap at κ^+ with a κ-Souslin tree

Theorem

Let κ be a regular uncountable cardinal, and let T be a streamlined, uniformly coherent κ-Souslin tree. Let $G \subseteq T$ be T-generic over V. Then:

1. $<\kappa$-$\mathrm{HOD}^{\mathrm{V}[G]} \subseteq \mathrm{V}$.
2. *If T is $<\kappa^+$-$\mathrm{HOD}^{V[G]}$, then $G \in <\kappa^+$-$\mathrm{HOD}^{\mathrm{V}[G]}$.*
3. *If $\bar{\kappa} \leq \kappa$ is a cardinal and T is $<\bar{\kappa}$-OD, then* $<\bar{\kappa}$-$\mathrm{HOD}^{\mathrm{V}[G]} \subseteq <\bar{\kappa}$-$\mathrm{HOD}^{\mathrm{V}}$.

It follows from recent work of Brodsky and Rinot that in L, for every regular cardinal κ that is not weakly compact, there is a streamlined, uniformly coherent κ-Souslin tree in L.

It follows from recent work of Brodsky and Rinot that in L, for every regular cardinal κ that is not weakly compact, there is a streamlined, uniformly coherent κ-Souslin tree in L.

Hence:

Corollary

Assume $\mathrm{V} = L$, and let λ be an uncountable regular cardinal that is not weakly compact. Then there is a λ-c.c. forcing extension $L[G]$ of L such that $\Lambda_0^{L[G]} = \lambda^+$.

It follows from recent work of Brodsky and Rinot that in L, for every regular cardinal κ that is not weakly compact, there is a streamlined, uniformly coherent κ-Souslin tree in L.

Hence:

Corollary

Assume $\mathrm{V} = L$, and let λ be an uncountable regular cardinal that is not weakly compact. Then there is a λ-c.c. forcing extension $L[G]$ of L such that $\Lambda_0^{L[G]} = \lambda^+$.

Corollary

If ZFC is consistent with the existence of an inaccessible cardinal, then it is consistent that Λ_0 is the successor of an inaccessible cardinal.

Iterating

Theorem

If ZFC is consistent with the existence of an inaccessible cardinal, then ZFC is consistent with the existence of a regular (in fact inaccessible) limit leap whose successor cardinal is also a leap.

Příkrý forcing

Theorem

Let κ be a measurable cardinal, let U be a normal ultrafilter on κ, let $\mathbb{P}$ be the Příkrý forcing for U, and let G be $\mathbb{P}$-generic over V. Then

1. $<\kappa$-$\mathrm{HOD}^{\mathrm{V}[G]} \subseteq \mathrm{V}$.
2. *If U is $<\kappa^+$-$\mathrm{OD}^{V[G]}$, then, letting C be the Příkrý sequence corresponding to G, $C \in <\kappa^+$-$\mathrm{HOD}^{\mathrm{V}[G]}$.*
3. *If $\bar{\kappa} \leq \kappa$ is a cardinal and U is $<\bar{\kappa}$-OD, then $<\bar{\kappa}$-$\mathrm{HOD}^{\mathrm{V}[G]} \subseteq <\bar{\kappa}$-$\mathrm{HOD}^{\mathrm{V}}$.*

Theorem

Assume $\mathrm{V} = L[U]$*, where* U *is a normal ultrafilter on* κ*. Let* $\mathbb{P}$ *be the Příkrý forcing for* U*, and let* G *be* $\mathbb{P}$*-generic over* V*. Then*

$$L[U] = \mathrm{HOD}^{L[U][G]} = {<}\kappa\text{-}\mathrm{HOD}^{L[U][G]} \subsetneq {<}\kappa^{+}\text{-}\mathrm{HOD}^{L[U][G]} = L[U][G].$$

In particular, $\Lambda_0 = \kappa^+$ *is the successor of a limit cardinal of countable cofinality in* $L[U][G]$*.*

Theorem

If ZFC is consistent with a measurable cardinal, then ZFC is also consistent with the existence of a singular limit leap of countable cofinality, whose cardinal successor is a leap.

Thank you for your attention!