Winter School 2023

January 28th–February 4th 2023
Štěkeň Chateau, Czech Republic

Invited speakers

- Clinton Conley
- Vera Fischer
- Aleksandra Kwiatkowska
- Assaf Rinot

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Introduction to big Ramsey degrees

David Chodounský

Let $\mathbf{A}$ be a countable structure and let F be a finite structure (in the same language). We say that F has a *finite big Ramsey degree* in $\mathbf{A}$ if there is a number $d(F) \in \omega$ such that for every finite coloring of $\binom{\mathbf{A}}{F}$ (copies of F in $\mathbf{A}$) there is $\mathbf{B} \in \binom{\mathbf{A}}{\mathbf{A}}$ (a copy $\mathbf{B}$ of $\mathbf{A}$ in $\mathbf{A}$) such that $\binom{\mathbf{B}}{F}$ has at most $d(F)$ colors.

We say that $\mathbf{A}$ has *finite big Ramsey degrees* if every finite F has a finite big Ramsey degree in $\mathbf{A}$.
Equivalently, if for every $n \in \omega$ there is $D(n) \in \omega$ such that for every finite coloring of $[\mathbf{A}]^n$ there is $\mathbf{B} \in \binom{\mathbf{A}}{\mathbf{A}}$ such that $[\mathbf{B}]^n$ has at most $D(n)$ colors.

Example

- $(\omega,$ no structure$)$ (Ramsey)
- $(\mathbb{Q}, <)$ (Galvin, Laver, Devlin)
- Random (Rado) graph (Todorčević, Sauer)
- Triangle free Henson graph $\mathbb{H}_3$ (Dobrinen, Hubička)
- Universal homogeneous poset (Hubička)
- Random 3-hypergraph (BHChKV)

Definition

A structure $\mathbf{A} \in \mathcal{C}$ is universal (for a class of structures $\mathcal{C}$) if $\mathbf{A}$ contains a copy of every $\mathbf{B} \in \mathcal{C}$.

Proposition

If $\mathbf{A}, \mathbf{B} \in \mathcal{C}$ are both universal for $\mathcal{C}$ and $\mathbf{A}$ has finite big Ramsey degrees, then $\mathbf{B}$ also has finite big Ramsey degrees.

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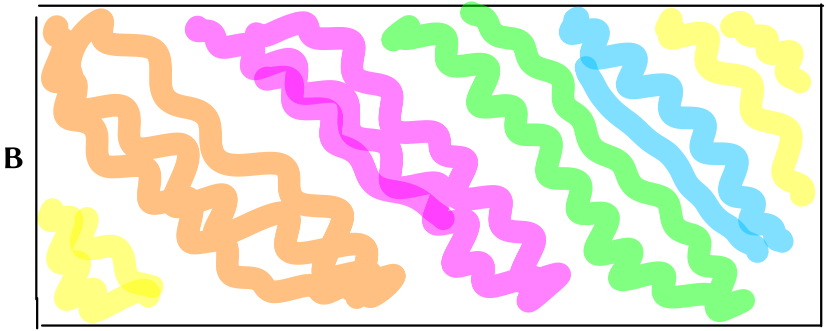

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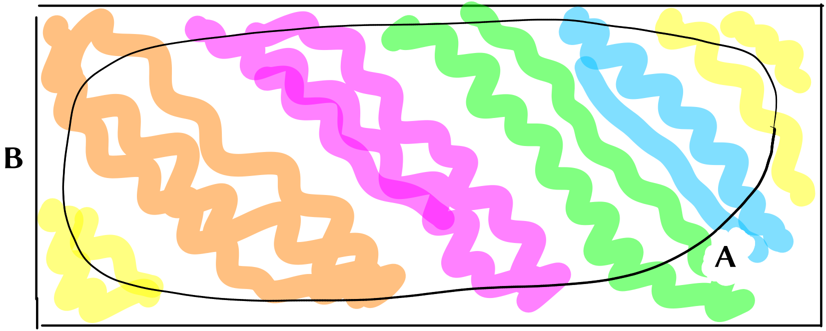

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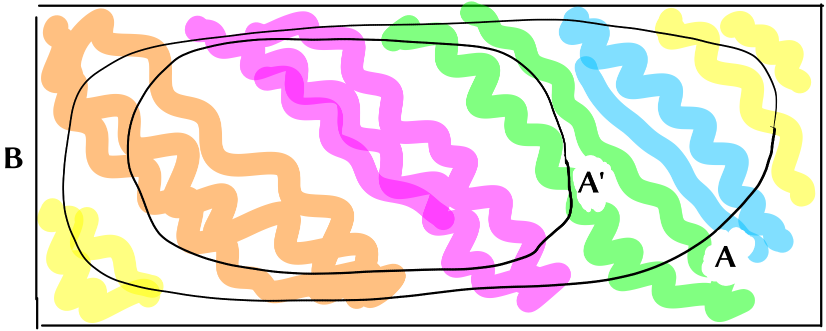

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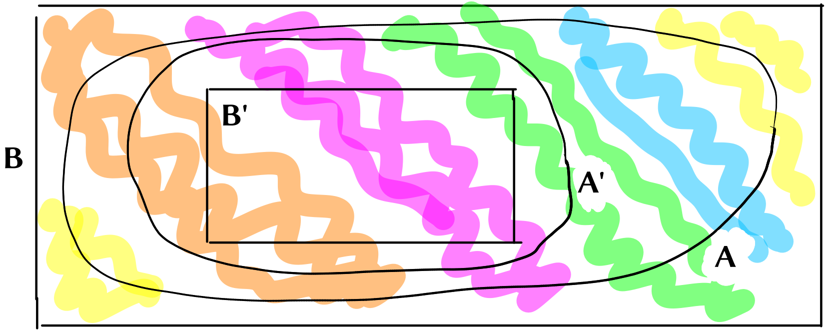

Trees

- rooted
- height at most ω ... $h(T) \leq \omega$
- finitely branching
- balanced (no short branches)
- n-th level of T ... $T(n)$
- initial subtree ... $T(<n)$
- set of immediate successors of s in T ... $isu_T(s)$

Definition

A subtree S of T of height $h(S) \in \omega + 1$ is a *strong subtree* if

- $\forall n < h(S)\ \exists m < h(T)$ such that $S(n) \subseteq T(m)$,
- $\forall s \in S\ \forall t \in isu_T(s)\ \exists!(s' \in S, s' \geq t, s' \in isu_S(s))$, unless $isu_S(s) = \emptyset$.

We write $S \in STR_n(T)$.

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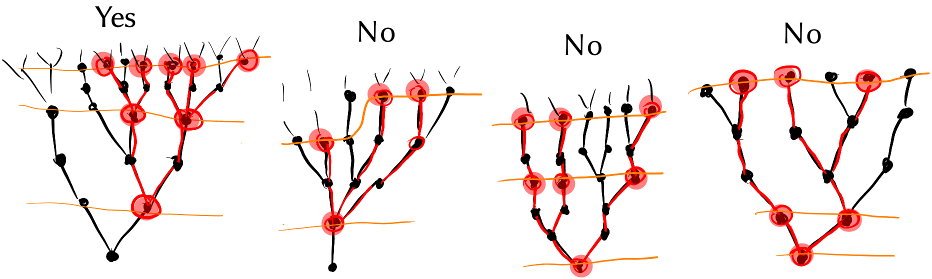

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If $S \in STR_n(T)$ and $R \in STR_m(S)$, then $R \in STR_m(T)$.

Theorem (Milliken, simple version)

If T is a tree of height ω, $n, k \in \omega$, and $\chi\colon\ STR_n(T) \to k$ is a finite coloring, then there exists $S \in STR_\omega(T)$ such that χ is monochromatic on $STR_n(S)$.

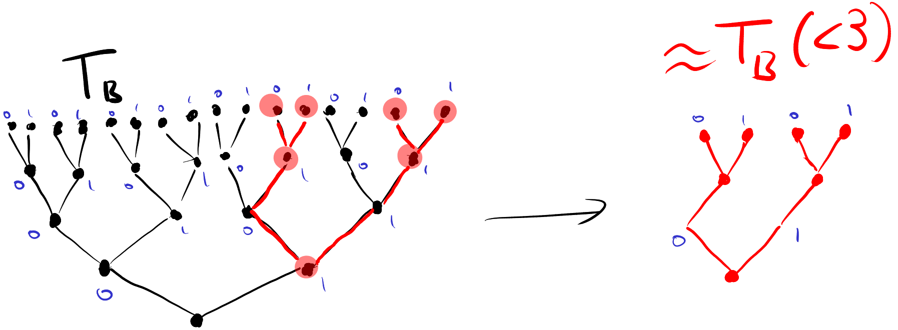
Trees, examples

Example

$\mathbf{T}_B = 2^{<\omega}$, the binary tree

Observation

If $S \in STR_n(\mathbf{T}_B)$, then S is isomorphic to $\mathbf{T}_B(<n)$.



Trees, examples

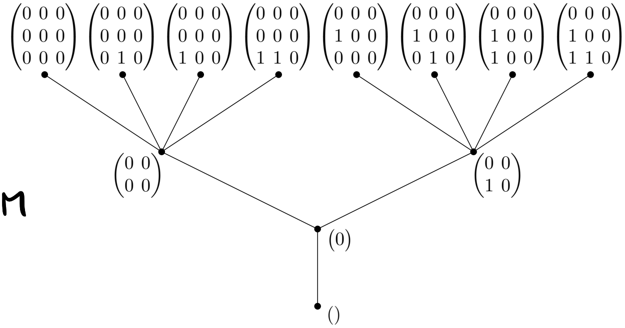
Example

$\mathbf{T}_M = \bigcup\{\, 2^{n\times n} : n \in \omega \,\}$, ordered by extension. The tree of matrices.

$\mathbf{T}_{SM} \subset \mathbf{T}_M$, the tree of sub-diagonal matrices.
If $A \in \mathbf{T}_{SM}$ and $A(i,j) \neq 0$, then $i < j$.

For $A \in \mathbf{T}_M(n)$ we write $|A| = n$.

$\mathbf{T}_{SM}$



Carlson–Simpson theorem

Σ ... finite alphabet (imagine $\{0, 1\}$)

Definition – k-parameter word

- $k \in \omega + 1$
- String (finite or countable) of symbols from $\Sigma \cup \{ \lambda_i \mid i \in k \}$.
- Each λ_i, $i \in k$ has to appear.
- If $i < j \in k$, then the first occurrence of λ_i comes before the first occurrence of λ_j.
- $[\Sigma](k)$ ... infinite k-parameter words.
- $[\Sigma]^*(k)$ ... finite k-parameter words.

Substitution

$W \in [\Sigma](n)$, $U \in [\Sigma]^*(k)$, $|U| \leq n \in \omega + 1$.

$W(U) \in [\Sigma]^*(k)$

- in W substitute $\lambda_i \mapsto U(i)$
- truncate at the first occurrence of $\lambda_{|U|}$

Carlson–Simpson theorem

Theorem

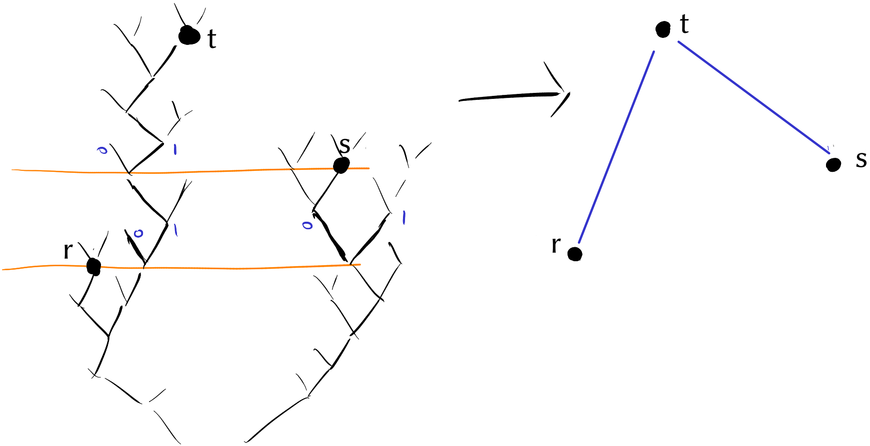
Let Σ be a finite alphabet, $k \in \omega$. If $[\Sigma]^*(k)$ is colored by finitely many colors, then there is $W \in [\Sigma](\omega)$ such that

$$W([\Sigma]^*(k)) = \{\, W(U) \mid U \in [\Sigma]^*(k) \,\}$$

is monochromatic.

Random graph has finite big Ramsey degrees

For $s, t \in \mathbf{T}_B$ define $E(s, t)$ if $|s| < |t|$ and $t(|s|) = 1$.



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(both as a graph and as a tree).

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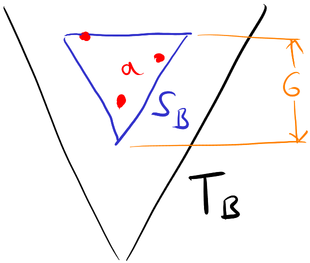
Lemma

For every $n \in \omega$ and $a \in [\mathbf{T}_B]^n$ there exists $S_B \in STR_{2n}(\mathbf{T}_B)$ such that $a \subset S_B$.



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S_B has size $2^{2n} - 1$. I.e. $[S_B]^n$ has size $< (2^{2n})^n$.

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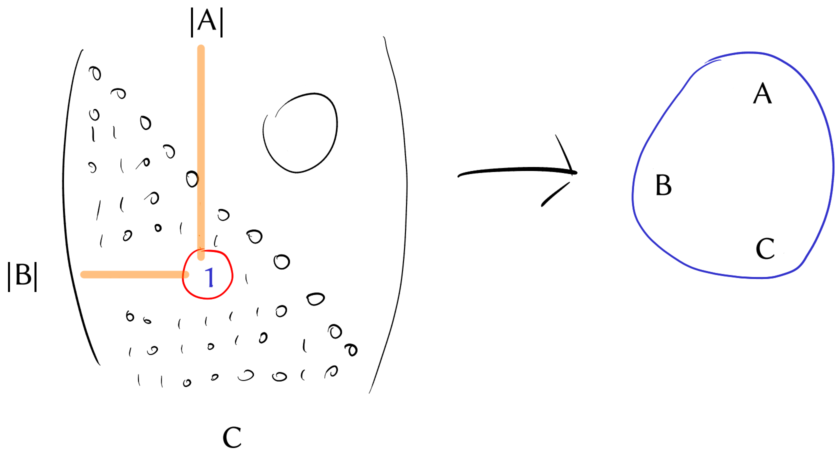
Proof

Given finite coloring $\chi\colon [\mathbf{T}_B]^n \to k$.
Induces finite coloring $\bar{\chi}\colon\ STR_{2n}(\mathbf{T}_B) \to k^{(2^{2n})^n}$.
Use Milliken's theorem to find $S \in STR_\omega(\mathbf{T}_B)$,
a $\bar{\chi}$-monochromatic copy of $\mathbf{T}_B$.

Universal 3-hypergraphs have finite big Ramsey degrees

For $A, B, C \in \mathbf{T}_{SM}$ define $E(A, B, C)$ is $|A| < |B| < |C|$ and $C(|A|, |B|) = 1$.



Universal 3-hypergraphs have finite big Ramsey degrees

For $A, B, C \in \mathbf{T}_{SM}$ define $E(A, B, C)$ is $|A| < |B| < |C|$ and $C(|A|, |B|) = 1$.

Proposition

The hypergraph $(\mathbf{T}_{SM}, E)$ *is universal (for countable 3-hypergraphs).*

Observation

If $S \in STR_\omega(\mathbf{T}_{SM})$, then (S, E) is not a copy of $(\mathbf{T}_{SM}, E)$.
(It is wider and we can find a copy of $\mathbf{T}_{SM}$ inside S.)

Problem

For $S \in STR_{2n}(\mathbf{T}_{SM})$ there is no bound on the size of S.
I.e. a finite coloring $\chi\colon [\mathbf{T}_{SM}]^n \to k$ does not induce a finite coloring $\bar{\chi}$ of $STR_{2n}(\mathbf{T}_{SM})$.

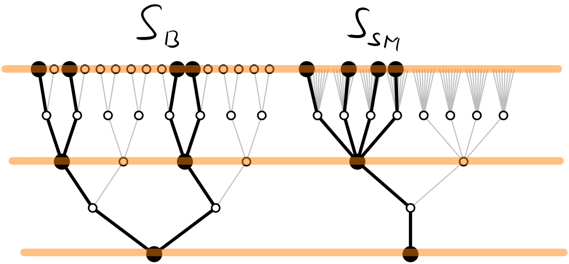
Product trees

$\mathbf{T}_{SM} \otimes \mathbf{T}_B$ ... the product tree

Definition

We say that $S_{SM} \otimes S_B \in STR_k(\mathbf{T}_{SM} \otimes \mathbf{T}_B)$ ($S_{SM} \otimes S_B$ is a strong subtree of $\mathbf{T}_{SM} \otimes \mathbf{T}_B$) if

- $S_{SM} \in STR_k(\mathbf{T}_{SM})$,
- $S_B \in STR_k(\mathbf{T}_B)$, and
- $\forall n \in k\ \exists m \in \omega$ such that $S_{SM}(n) \subseteq \mathbf{T}_{SM}(m)$ and $S_B(n) \subseteq \mathbf{T}_B(m)$.



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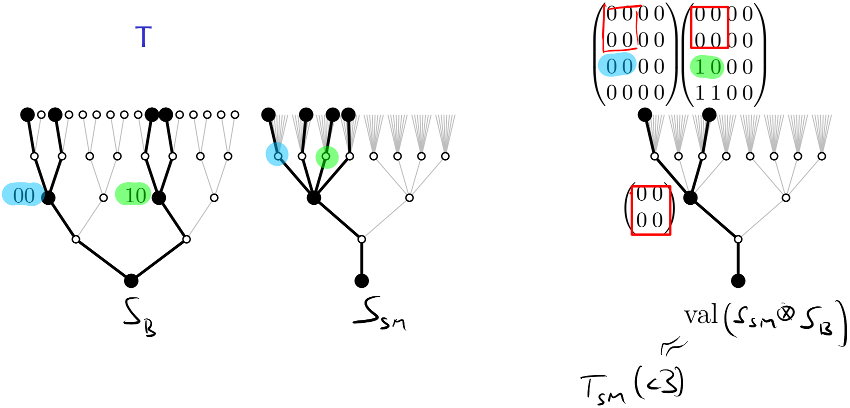
Theorem (Milliken, special case)

If $n, k \in \omega$ and $\chi : \ STR_n(\mathbf{T}_{SM} \otimes \mathbf{T}_B) \to k$ is a finite coloring, then there exists $S_{SM} \otimes S_B \in STR_\omega(\mathbf{T}_{SM} \otimes \mathbf{T}_B)$ such that χ is monochromatic on $STR_n(S_{SM} \otimes S_B)$.

Valuations

Suppose $S_{SM} \otimes S_B \in STR_k(\mathbf{T}_{SM} \otimes \mathbf{T}_B)$ for some $k \in \omega + 1$.
We define the tree $val(S_{SM} \otimes S_B) \subseteq S_{SM}$ by induction:

- The root of $val(S_{SM} \otimes S_B)$ is the root of S_{SM}.
- If $A \in val(S_{SM} \otimes S_B)$, $t \in S_B(|A|)$, $C \in isu_{S_{SM}}(A)$, and $C > A^\frown t$, then $C \in val(S_{SM} \otimes S_B)$.



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Observation

If $S_{SM} \otimes S_B \in STR_k(\mathbf{T}_{SM} \otimes \mathbf{T}_B)$,
then $(val(S_{SM} \otimes S_B), E)$ is a copy of $(\mathbf{T}_{SM}(< k), E)$
(both as a hypergraph and as a tree).

Lemma (false but fixable)

For every $n \in \omega$ and $a \in [\mathbf{T}_{SM}]^n$ there exists
$S_{SM} \otimes S_B \in STR_{2n}(\mathbf{T}_{SM} \otimes \mathbf{T}_B)$ such that $a \subset val(S_{SM} \otimes S_B)$.

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Proof

Given finite coloring $\chi\colon [\mathbf{T}_{SM}]^n \to k$.
Induces finite coloring $\bar{\chi}\colon\ STR_N(\mathbf{T}_{SM} \otimes \mathbf{T}_B) \to K$
(look at colors on valuations).
Use Milliken's theorem.