Ordinal Ramsey Theory

Thilo Weinert
University of Vienna

Set Theory Research Seminar University of Vienna, Tuesday, 1^{st} March 2022

Definition

$\beta \longrightarrow (\gamma, \delta)^2$ means that every graph on a set of size β has an independent set of size γ or a complete subgraph of size δ.

Definition

$r(\gamma, \delta) = \beta$ means $\beta \longrightarrow (\gamma, \delta)^2$ but $\zeta \not\longrightarrow (\gamma, \delta)^2$ for all $\zeta < \beta$.

Example

$r(3, 3) = 6$.

Notation

For a graph G let

- $n = n_G$ *be the number of its vertices,*
- $e = e_G$ *be the number of its edges and*
- $d = d_G = \frac{2e_G}{n_G}$ *be its average degree.*
- $d^{max} = d_G^{max}$ *be its maximum degree.*
- $\alpha = \alpha_G$ *the minimal size of an independent set.*

Theorem (Turán, ?)

$$\alpha \geqslant \frac{n}{d+1}.$$

Observation

For triangle-free graphs, $\alpha \geqslant d$.

Corollary

$n(n+1) \longrightarrow (n,3)^2$.

Theorem (Erdős, 1961)

There is a constant $c > 0$ *such that* $\left\lfloor \frac{cn^2}{(\ln(n))^2} \right\rfloor \not\longrightarrow (n,3)^2$ *for all natural numbers* n.

Theorem (Graver & Yackel, 1968)

There is a constant $c > 0$ *such that* $\left\lfloor \frac{cn^2 \ln(\ln(n))}{\ln(n)} \right\rfloor \longrightarrow (n,3)^2$ *for all natural numbers* n.

Theorem (Ajtai, Komlós & Szemerédi, 1980)

There is a constant $c > 0$ *such that* $\left\lfloor \frac{cn^2}{\ln(n)} \right\rfloor \longrightarrow (n, 3)^2$ *for all* $n \in \omega \setminus 2$.

Theorem (Shearer, 1982)

$\alpha \geqslant \frac{n(d\ln(d) + 1 - d)}{(d-1)^2}$ *for triangle-free graphs.*

Corollary

An version of the Theorem of Ajtai, Komlós, and Szemerédi with smaller c.

Theorem (Kim, 1995)

There is a constant $c > 0$ *such that* $\left\lfloor \frac{cn^2}{\ln(n)} \right\rfloor \not\longrightarrow (n,3)^2$ *for all* $n \in \omega \setminus 2$.

Corollary

There is a constant $c > 0$ *such that* $\left\lfloor \frac{cn^2}{\ln(n)} \right\rfloor \not\longrightarrow (I_n, L_3)^2$ *for all* $n \in \omega \setminus 2$.

Notation

$k \longrightarrow (I_m, L_n)^2$ *if and only if every oriented graph on a set of size* k *has an independent set of size* m *or a complete cyclefree subgraph of size* n.

Theorem (Erdős & Rado for $\kappa = \omega$, Baumgartner for cardinals $\kappa > \omega$)

$\kappa k \longrightarrow (\kappa m, n)^2$ *if and only if* $k \longrightarrow (I_m, L_n)^2$ *for all infinite cardinals* κ.

Theorem (Ramsey's Theorem for two colours)

$\omega \longrightarrow (\omega, \omega)^n$ *for every natural number* n.

Definition

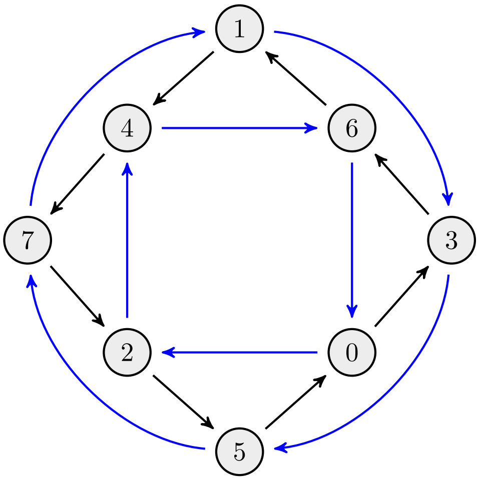
$r(I_k, L_m) = n$ means $n \longrightarrow (I_k, L_m)^2$ but $p \not\longrightarrow (I_k, L_m)^2$ for all $p < n$.

Example (Erdős & Rado, 1956)

$r(I_2, L_3) = 4$.

Example (Bermond, 1974)

$8 \not\longrightarrow (I_3, L_3)^2$.



$x \mapsto x + 3$

$x \mapsto x + 2.$

Theorem (Stearns, 1959)
$2^{n-1} \longrightarrow (I_2, L_n)^2$ *for all natural numbers* n.

Example (Larson & Mitchell, 1997)
$13 \not\longrightarrow (I_4, L_3)^2$.

Theorem (Larson & Mitchell, 1997)
$n^2 \longrightarrow (I_n, L_3)^2$ *for all natural numbers* n.

Theorem (Ihringer, Rajendraprasad & W.)
$n^2 - n + 3 \longrightarrow (I_n, L_3)^2$ *for* $n \in \omega \setminus 2$.

Lemma

$r(I_{n+1}, L_{m+l}) < 2r(I_{n+1}, L_m) + r(I_n, L_{m+1}) + 1.$

Proposition

We have $r(I_m, L_n) \leqslant v(m, n)$ *for all natural numbers* m *and* n *with* $m \geqslant 2$ *and* $n \geqslant 3$ *where*

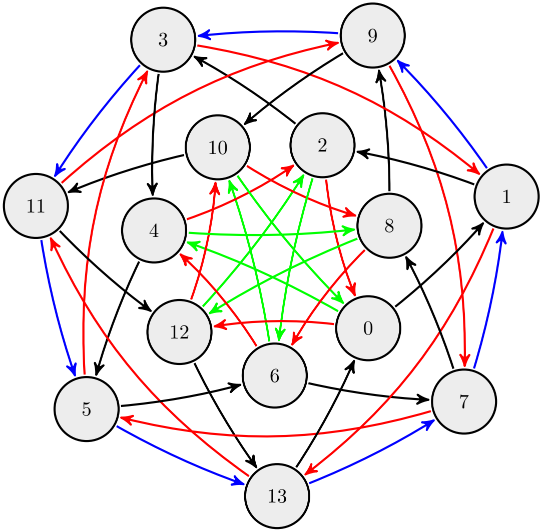
$$v(m,n) := \sum_{i=0}^{n-2} \binom{i+m-1}{i+1} 2^i - \binom{m+n-6}{m-4} 2^{n-3} + 1.$$

Example (Rajendraprasad)

$14 \not\to (I_4, L_3)^2.$

Corollary

$r(I_4, L_3) = 15.$



$x \mapsto x + 1$

$x \mapsto x - 2$

$x \mapsto x{+}4$ if x is even

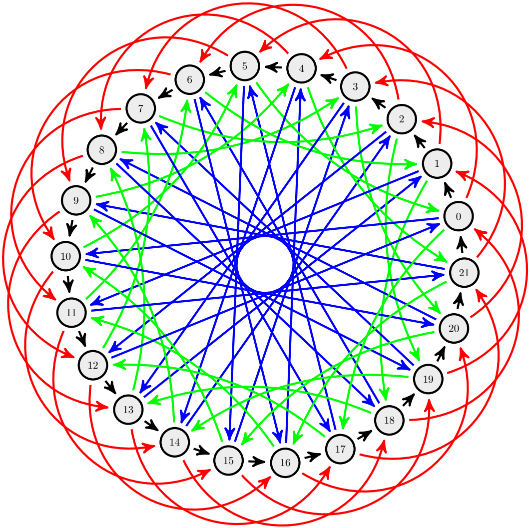
$x \mapsto x{-}6$ if x is odd.

Example

$22 \not\rightarrow (I_5, L_3)^2$.

Corollary

$r(I_4, L_3) = 23$.



$x \mapsto x+1$ (black)

$x \mapsto x+4$ (red)

$x \mapsto x-5$ (green)

$x \mapsto x+10$ (blue)

Theorem (Alon, 1996)

Considering a graph with at least one edge in which the neighbourhood of any vertex is r-colourable, we have
$$\alpha \geqslant \frac{n \operatorname{ld}(d^{max})}{160 d^{max} \operatorname{ld}(r+1)}.$$

Corollary

$$\left\lfloor \frac{508n^2}{\operatorname{ld}(n)} \right\rfloor \longrightarrow (I_n, L_3)^2.$$

Lemma (Alon, 1996)

Let $\mathcal{F}$ be a family of k distinct subsets of an n-element set X . Then the average size of a member of $\mathcal{F}$ is at least

$$\frac{\operatorname{ld}(k)}{10\operatorname{ld}\left(\frac{\operatorname{ld}(k)+n}{\operatorname{ld}(k)}\right)}.$$

Lemma (Tentative Improvement, Almost Proven)

Let $\mathcal{F}$ be a family of k distinct subsets of an n-element set X . Then the average size of a member of $\mathcal{F}$ is at least

$$\frac{(3-\sqrt{8})\operatorname{ld}(k)}{\operatorname{ld}\left(\frac{\operatorname{ld}(k)+n}{\operatorname{ld}(k)}\right)}.$$

Note that $3-\sqrt{8} > \frac{1}{6}$.

The Lemma would yield the following:

Proposition (Almost Proven)

Considering a graph with at least one edge in which the neighbourhood of any vertex is 2-colourable, we have $\alpha \geqslant \frac{n \operatorname{ld}(d^{max})}{13 d^{max}}$.

Corollary (Almost Proven)

$\left\lfloor \frac{26n^2}{\operatorname{ld}(n)} \right\rfloor \longrightarrow (I_n, L_3)^2$ *for all natural numbers* n.

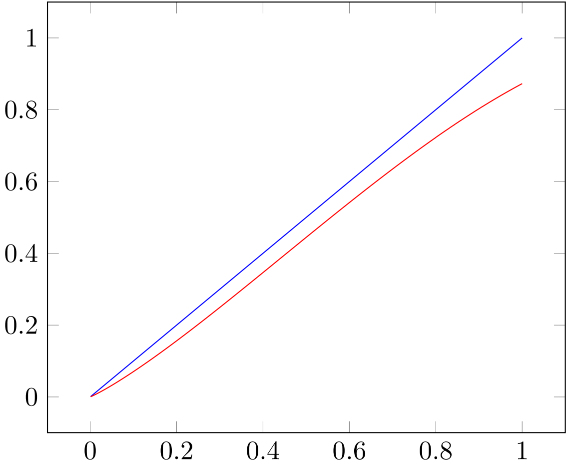
This all hinges on proving the seemingly true inequality

$$H\left(\frac{(2-\sqrt{2})\,x}{2\,\mathrm{ld}\left(1+\frac{1}{x}\right)}\right) \leqslant x \text{ for all } x \in [0,1]$$

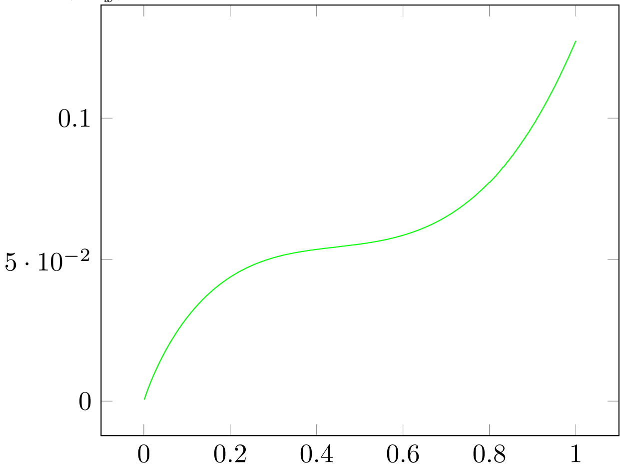
where H is the binary entropy function

$$\begin{aligned} H :]0,1[&\longrightarrow \mathbb{R} \\ x &\longmapsto -\,\mathrm{ld}\,(x)\,x - \mathrm{ld}\,(1-x)\,(1-x) \end{aligned}$$

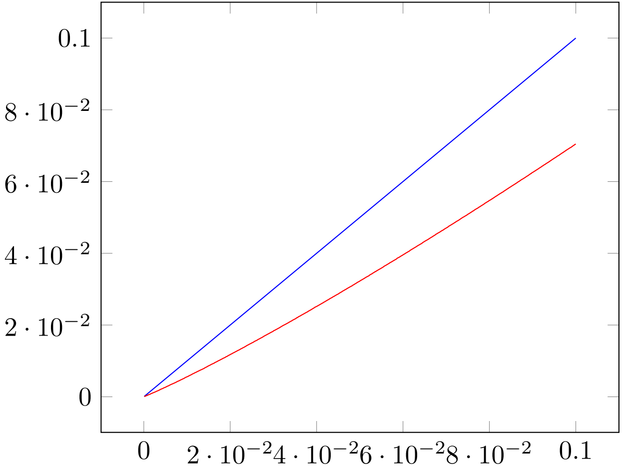
$H(\frac{2-\sqrt{2})x}{2\,\mathrm{ld}(1+\frac{1}{x})})$ and x



$H\left(\frac{(2-\sqrt{2})x}{2\,\mathrm{ld}(1+\frac{1}{x})}\right) - x$



$H(\frac{2-\sqrt{2})x}{2\,\mathrm{ld}(1+\frac{1}{x})})$ and x, closer to 0.



	3	4	5	6	7	8	9	m
3	6	9	14	18	23	28	36	
4	9	18	25					
5	14	25						
6	18							
7	23							
8	28							
9	36							
n								

Observation
$r(n+1,3) - r(n,3) \leqslant n+1.$

Proof.
Fix a vertex v in a graph on $r(n,3)+n+1$ vertices. Then either v has a neighbourhood of $n+1$ vertices or v is independent from a set of size $r(n,3)$. □

Proposition (Graver & Yackel, 1968)

Let G be a $(3, y)$-graph on n points with e edges. Let p_1 and p_2 be two points of G a distance of at least 5 apart (i.e., any path joining p_1 and p_2 has at least 5 edges). Denote the valence of p_i by $v_i (i = 1, 2)$; and let K_i represent the v_i points which are adjacent to p_i. Finally let G' be the graph formed by removing from G the points p_1 and p_2 and all edges with p_1 or p_2 as end-points, and then adding all edges between points in K_1 and points in K_2. Then G' is a $(3, y - 1)$-graph on $(n - 2)$ points with $[e + (v_1 - 1)(v_2 - 1) - 1]$ edges

Corollary

$r(n + 1, 3) - r(n, 3) \geqslant 3$ *for all* $n \in \omega \setminus 2$.

Conjecture (Erdős & Sós)

$\liminf_{n \nearrow \infty} r(n + 1, 3) - r(n, 3) = \infty$.

$r(I_m, L_n)$.

	2	3	4	5	m
3	4	9	15	23	
4	8	?			
5	14				
6	28				
n					

Observation

$r(I_{n+1}, L_3) - r(I_n, L_3) \leqslant 2n + 1.$

Proof.

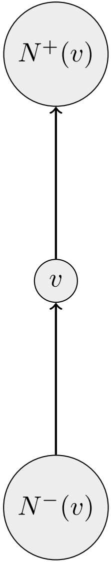
Fix a vertex v in a graph on $r(I_n, L_3) + 2n + 1$ vertices. Then either v has an in-neighbourhood of $n + 1$ vertices or an out-neighbourhood of $n + 1$ vertices or v is independent from a set of size $r(I_n, L_3)$. □

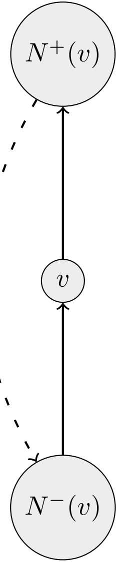
Proposition

Let e, i, and n be natural numbers. If there is an oriented graph all whose triangles are cyclic and all whose independent sets are smaller than i,with e edges on n vertices one of which is v having degree d, then there is an oriented graph on $n + 5$ vertices with $2d + e + 9$ edges all whose triangles are cyclic and all whose independent sets have size at most i.

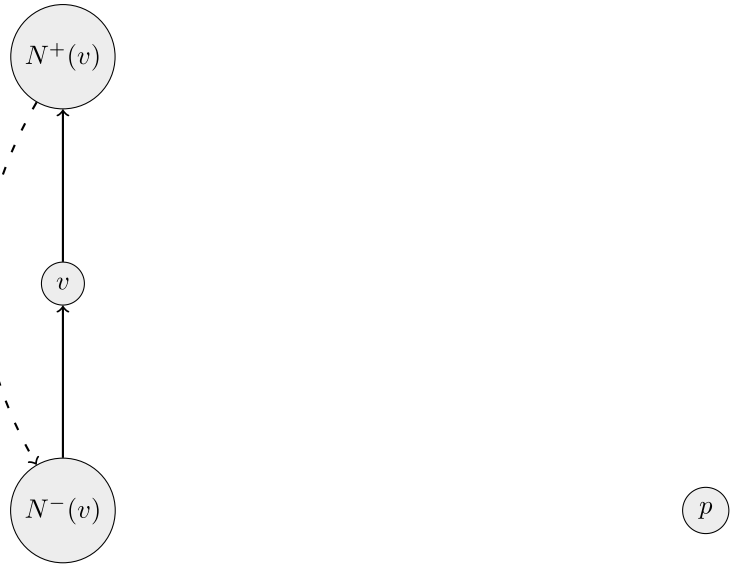
Corollary

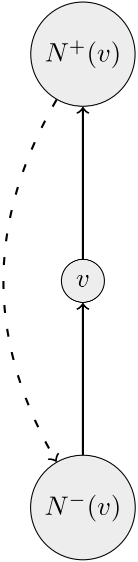
$r(I_{n+1}, L_3) \geqslant r(I_n, L_3) + 5$ *for all* $n \in \omega \setminus 2$.

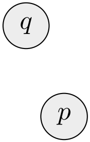

$N^+(v)$
v
$N^-(v)$



$N^+(v)$
v
$N^-(v)$


$N^+(v)$
v
$N^-(v)$
p



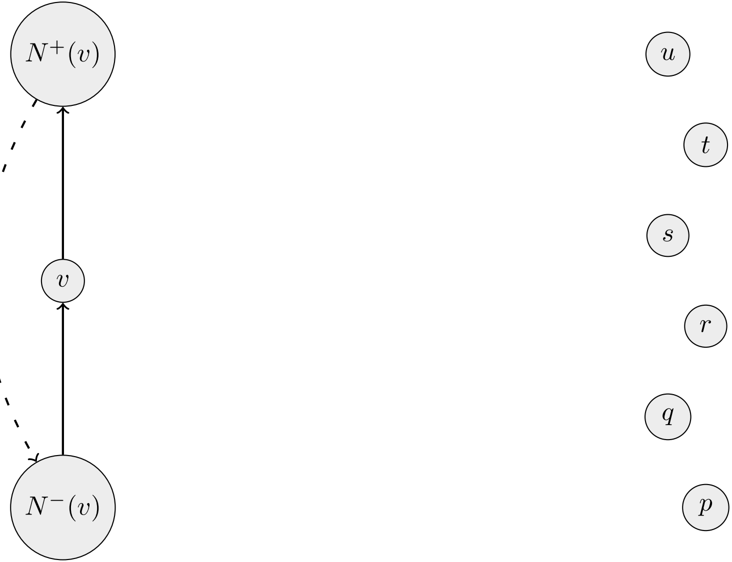


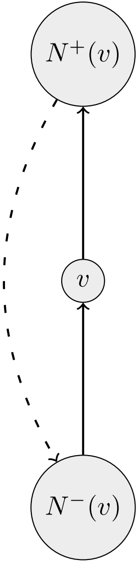
Thilo Weinert
University of Vienna

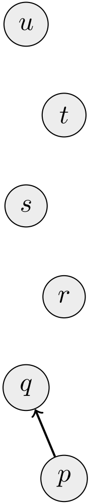

$N^+(v)$
v
$N^-(v)$
r
q
p


$N^+(v)$
v
$N^-(v)$
s
r
q
p


$N^+(v)$
v
$N^-(v)$
t
s
r
q
p

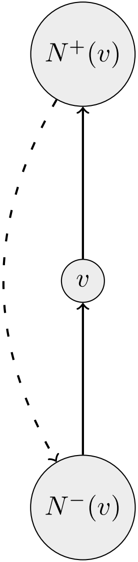

$N^+(v)$
v
$N^-(v)$
u
t
s
r
q
p

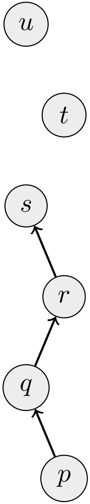




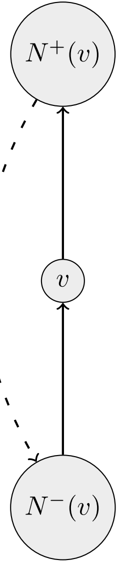
Thilo Weinert
University of Vienna

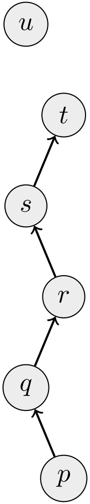

$N^+(v)$
v
$N^-(v)$
u
t
s
r
q
p



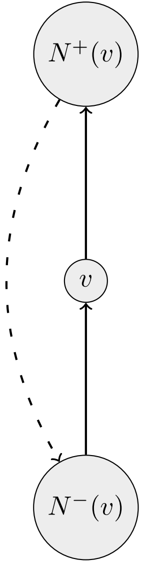


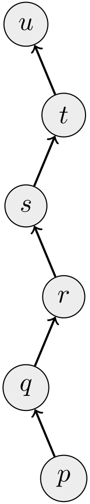
Thilo Weinert
University of Vienna



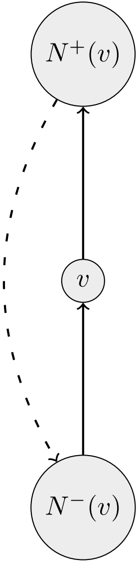


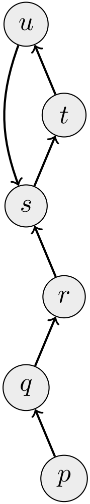
Thilo Weinert
University of Vienna

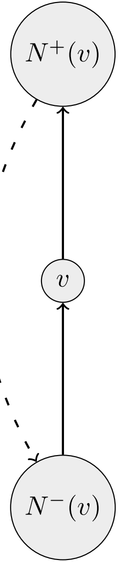




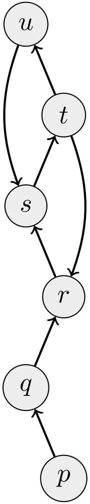
Thilo Weinert
University of Vienna


$N^+(v)$
v
$N^-(v)$

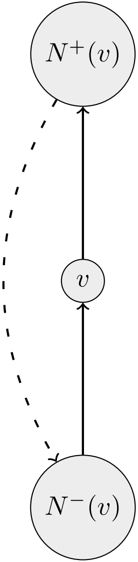

u
t
s
r
q
p

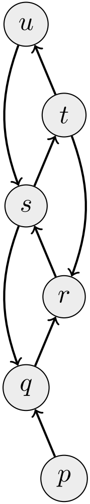


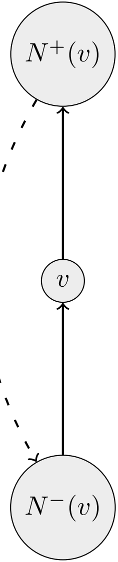
$N^+(v)$
v
$N^-(v)$



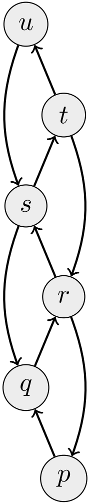
u
t
s
r
q
p


$N^+(v)$
v
$N^-(v)$

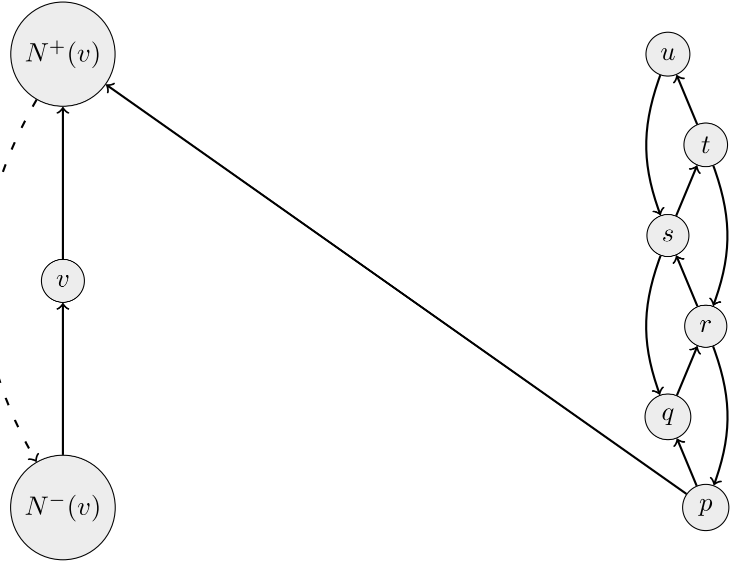

u
t
s
r
q
p

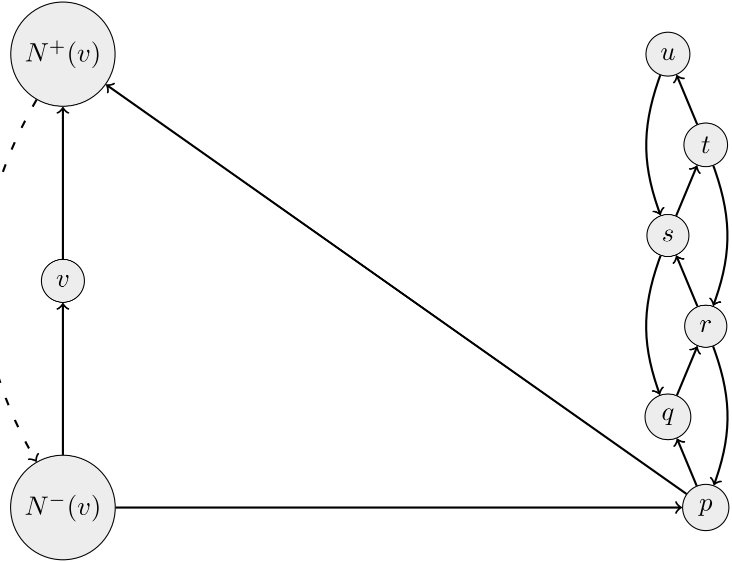


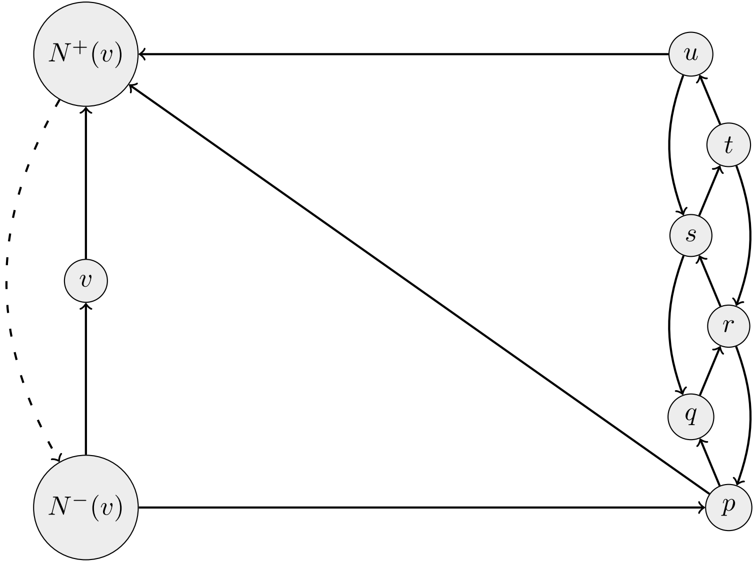
$N^+(v)$
v
$N^-(v)$



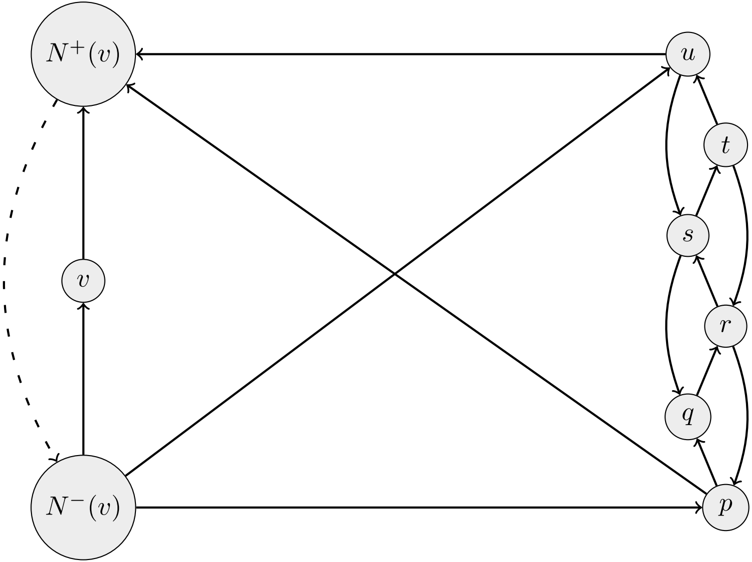
u
t
s
r
q
p


$N^+(v)$
v
$N^-(v)$
u
t
s
r
q
p

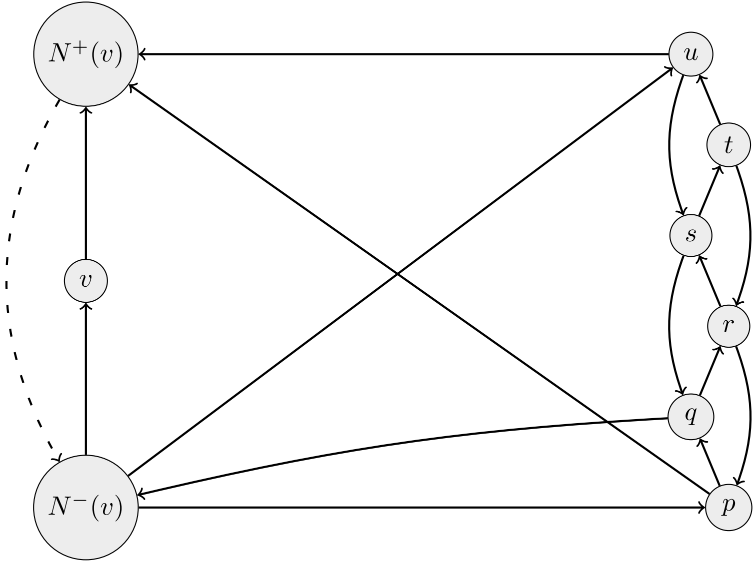

$N^+(v)$
v
$N^-(v)$
u
t
s
r
q
p

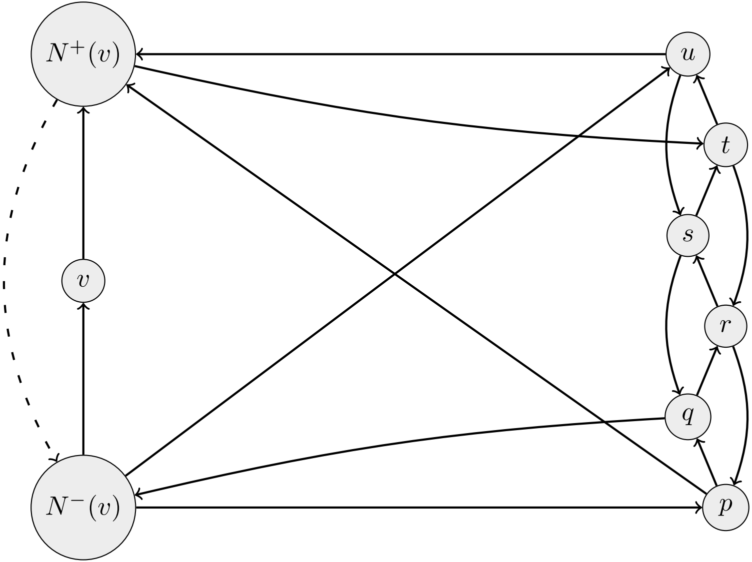


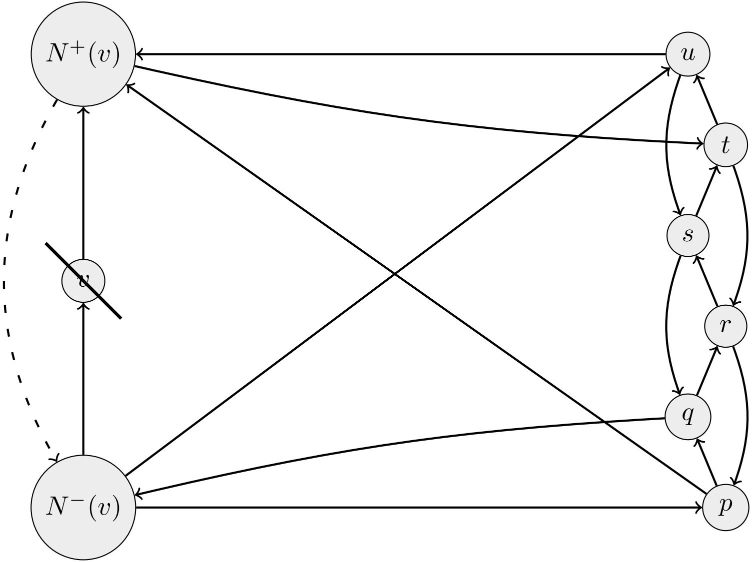
$N^+(v)$
u
t
s
v
r
q
$N^-(v)$
p

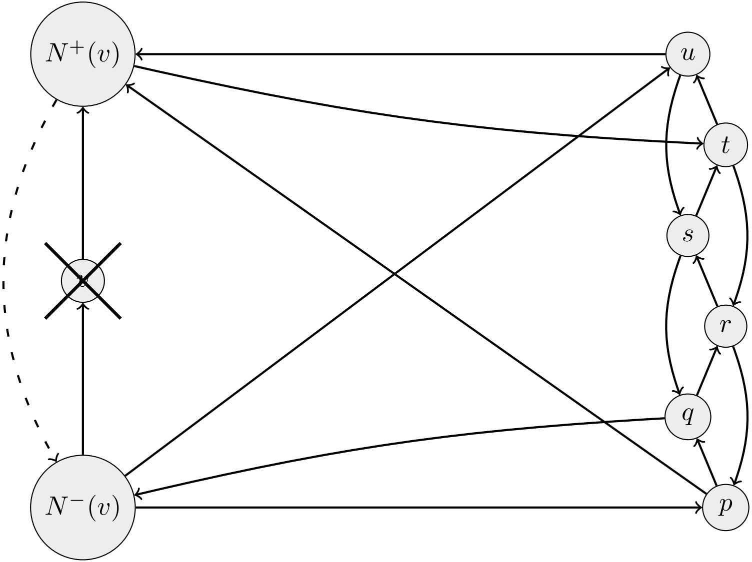


$N^+(v)$
u
t
s
v
r
q
$N^-(v)$
p


$N^+(v)$
u
t
s
v
r
q
$N^-(v)$
p


$N^+(v)$
u
t
s
v
r
q
$N^-(v)$
p


$N^+(v)$
u
t
s
v
r
q
$N^-(v)$
p

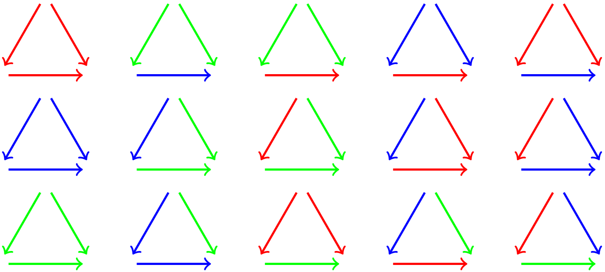


Ordinal Ramsey Theory

Thilo Weinert
University of Vienna

Definition

We identify 0 with red, 1 with blue and 2 with green arrows.



Theorem

For $i \in 3 \setminus 1$ *and* $\kappa \in \Omega \setminus 3$ *such that* $\kappa \longrightarrow (\kappa)^2_2$ *the partition relation* $\kappa^i l \longrightarrow (\kappa^i m, n)$ *holds true if and only if every coloured digraph* $C = \langle l, A, c \rangle$ *with* $\mathrm{ran}(c) = 2i - 1$ *contains an independent set of size* m *or there is a subtournament* S *of* C *induced by a set of* n *vertices such that all triples in* S *are agreeable.*

Lemma

$r(I_{n+1}, A_3) \leqslant r(I_n, A_3) + 2r(I_{n+1}, L_3) + 4n - 1$ *for all* $n \in \omega \setminus 2$.

Theorem

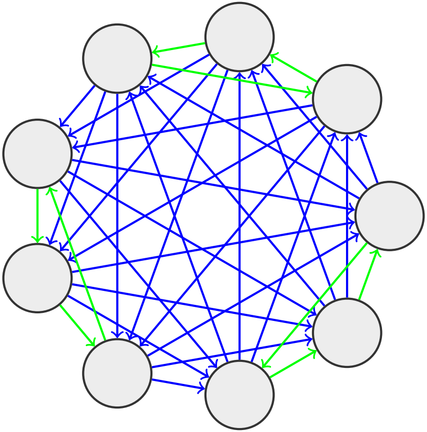
For all $n \in \omega \setminus 2$ *we have*

$$r(I_n, A_3) \leqslant \frac{(2n+1)(n^2+4n-6)}{3}. \tag{1}$$

Theorem

$r(I_2, A_3) = 10.$

Proof.



□

Theorem (Hajnal, 1971)

The continuum hypothesis implies $\omega_1^2 \not\longrightarrow (\omega_1^2, 3)^2$.

Theorem (Erdős & Hajnal, 1973)

The continuum hypothesis implies $\omega_1\omega \not\longrightarrow (\omega_1\omega, 3)^2$.

Theorem (Baumgartner, 1975)

$\omega_1^2 \longrightarrow (\omega_1^2, 3)^2$ *implies the Souslin hypothesis.*

Theorem (Takahashi, 1987)

$\text{stick} = \aleph_1$ *implies* $\omega_1^2 \not\longrightarrow (\omega_1^2, 3)^2$.

Theorem (Takahashi, 1987)

$\mathfrak{d} = \aleph_1 = \text{stick}$ *implies* $\omega_1\omega \not\longrightarrow (\omega_1\omega, 3)^2$.

Theorem (J. Larson, 1998)

$\mathfrak{d} = \aleph_1$ *implies* $\omega_1^2 \not\longrightarrow (\omega_1^2, 3)^2$.

Theorem (J. Larson, 1998)

$\mathfrak{d} = \aleph_1$ *implies* $\omega_1\omega \not\longrightarrow (\omega_1\omega, 3)^2$.

Theorem (Lambie-Hanson & W., 2016)

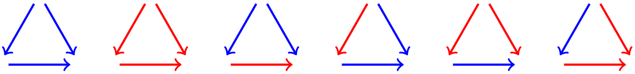
$\mathfrak{b} = \aleph_1 = \text{⫯}$ *implies* $\omega_1\omega \not\longrightarrow (\omega_1\omega, 3)^2$.

Theorem (Baumgartner, 1989)

$\mathrm{MA}_{\aleph_1}$ *implies* $\omega_1\omega \longrightarrow (\omega_1\omega, n)^2$ *for all natural numbers* n.

Definition

A triple is called *strongly agreeable* if and only if it is agreeable and does not contain any green arrow. So it is strongly agreeable precisely if it is one of these:



Theorem

$r(\kappa\lambda m, n) = \kappa\lambda r(I_m, S_n)$ for κ weakly compact and any cardinal $\lambda \in \kappa \setminus \omega$.

Theorem

If $\mathrm{MA}_{\aleph_1}$ holds true then $r(\omega_1\omega m, n) = \omega_1\omega r(I_m, S_n)$.

Lemma

$r(I_{m+1}, S_3) \leqslant r(I_m, S_3) + 4m + 1$ for all $m \in \omega \setminus 2$.

Theorem

For all $m \in \omega \setminus 2$ we have $r(I_m, S_3) \leqslant m(2m-1)$.

Lemma

For all $n \in \omega \setminus 3$ we have $r(I_2, S_{n+1}) \leqslant 4r(I_2, S_n) - 2$.

Theorem

For any $n \in \omega \setminus 3$ we have

$$r(I_2, S_n) \leqslant \frac{4^{n-1}+2}{3}.$$

Lemma

$r(I_{m+1}, S_{n+1}) \leqslant r(I_m, S_{n+1}) + 4r(I_{m+1}, S_n) - 3$ for all $m \in \omega \setminus 2$ and all $n \in \omega \setminus 3$.

Theorem

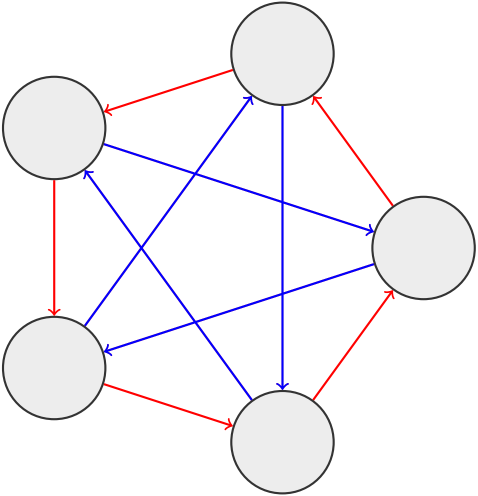
For all $m \in \omega \setminus 2$ and all $n \in \omega \setminus 3$ we have $r(I_m, S_n) \leqslant u(m, n)$ where

$$u(m,n) := \frac{1}{4}\left(3 + \sum_{i=0}^{n-1} \binom{i+m-2}{i} 4^i\right). \tag{2}$$

Theorem

$r(I_2, S_3) = 6.$

Proof.

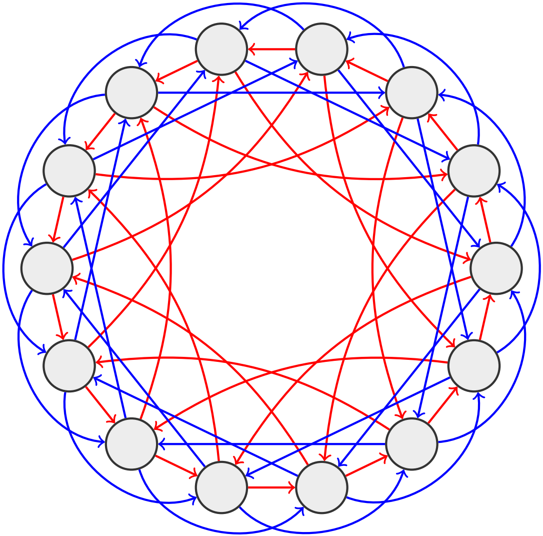


□

Theorem

$r(I_3, S_3) = 15.$

Proof.



□

Theorem (Specker, 1956)
$\omega^2 \longrightarrow (\omega^2, n)^2$ *for every natural number* n.

Theorem (Specker, 1956)
$\omega^3 \not\longrightarrow (\omega^3, 3)^2$.

Theorem (Nosal, 1972)
$r(\omega^3, n) = \omega^{\lfloor \mathrm{ld}(n) \rfloor + 2}$.

Theorem (Nosal, 1976)
$r(\omega^m, n) = \omega^{1+\lfloor \mathrm{ld}(n) \rfloor (m-1)}$ *for* $m \in \omega \setminus 5$.

Theorem (Chang & Milner)
$\omega^\omega \longrightarrow (\omega^\omega, n)^2$ *for all natural numbers* n.

Theorem (Chang, Darby & Schipperus)

If the additive normal form of $\beta < \omega_1$ has one or two summands, then $\omega^{\omega^\beta} \longrightarrow (\omega^{\omega^\beta}, 3)^2$.

Theorem (Darby, Schipperus, Larson)

If $\beta \geqslant \gamma \geqslant 1$*, then* $\omega^{\omega^{\beta+\gamma}} \not\longrightarrow (\omega^{\omega^{\beta+\gamma}}, 5)^2$.

Theorem (Darby & J. Larson)

$\omega^{\omega^2} \longrightarrow (\omega^{\omega^2}, 4)^2$.

Question

What is $r(I_3, L_4)^2$?

We know that $r(I_3, L_4) \in 25 \setminus 21 = \{21, 22, 23, 24\}$.
For context:

Theorem (Codish, Frank, Itzhakov & Miller, 2016)

$r(3, 3, 4) = 30$.

Question

What is the order of growth of $r(I_n, A_3)$?

Question

$\liminf_{n \nearrow \infty} r(I_{n+1}, L_3) - r(I_n, L_3) = \infty$?

Question

$\liminf_{n \nearrow \infty} r(I_{n+1}, S_3) - r(I_n, S_3) = \infty$?

Question

$\liminf_{n \nearrow \infty} r(I_{n+1}, A_3) - r(I_n, A_3) = \infty$?

Question

Can Nosal's theorem be extended to $r(\omega^m, n) = \omega^{1+\lfloor \mathrm{ld}(n) \rfloor (m-1)}$ for $m \in \omega \setminus 4$?

Question

$\omega^{\omega^3} \longrightarrow (\omega^{\omega^3}, 3)^2$*?*

Question

Is $\omega_1^2 \longrightarrow (\omega_1^2, 3)^2$ consistent?

Miklós Ajtai, János Komlós, and Endre Szemerédi. A note on Ramsey numbers. *J. Combin. Theory Ser. A*, 29(3):354–360, 1980. ISSN 0097-3165. doi:10.1016/0097-3165(80)90030-8. URL https://doi.org/10.1016/0097-3165(80)90030-8.

Noga Alon. Independence numbers of locally sparse graphs and a Ramsey type problem. *Random Structures Algorithms*, 9 (3):271–278, 1996. ISSN 1042-9832. doi:10.1002/(SICI)1098-2418(199610)9:3<271::AID-RSA1>3.0.CO;2-U. URL http://dx.doi.org/10.1002/(SICI)1098-2418(199610)9:3<271::AID-RSA1>3.0.CO;2-U.

James Earl Baumgartner. Improvement of a partition theorem of Erdös and Rado. *J. Combinatorial Theory Ser. A*, 17: 134–137, 1974.

Jean-Claude Bermond. Some Ramsey numbers for directed graphs. *Discrete Math.*, 9:313–321, 1974. ISSN 0012-365X.

Fan Rong K Chung Graham and Ronald Lewis Graham. *Erdős on graphs*. A K Peters Ltd., Wellesley, MA, 1998. ISBN 1-56881-079-2; 1-56881-111-X. His legacy of unsolved problems.

Michael Codish, Michael Frank, Avraham Itzhakov, and Alice Miller. Computing the Ramsey number $R(4, 3, 3)$ using abstraction and symmetry breaking. *Constraints*, 21(3):375–393, 2016. ISSN 1383-7133. doi:10.1007/s10601-016-9240-3. URL https://doi.org/10.1007/s10601-016-9240-3.

Paul Erdős. Graph theory and probability. II. *Canadian J. Math.*, 13:346–352, 1961. ISSN 0008-414X. doi:10.4153/CJM-1961-029-9. URL https://doi.org/10.4153/CJM-1961-029-9.

Paul Erdős. Some new problems and results in graph theory and other branches of combinatorial mathematics. In *Combinatorics and graph theory (Calcutta, 1980)*, volume 885 of *Lecture Notes in Math.*, pages 9–17. Springer, Berlin-New York, 1981.

Paul Erdős and Richard Rado. A partition calculus in set theory. *Bull. Amer. Math. Soc.*, 62:427–489, 1956. ISSN 0002-9904. URL http://www.ams.org/journals/bull/1956-62-05/S0002-9904-1956-10036-0/S0002-9904-1956-10036-0.pdf.

Paul Erdős and Leo Moser. On the representation of directed graphs as unions of orderings. *Magyar Tud. Akad. Mat. Kutató Int. Közl.*, 9:125–132, 1964. URL http://www.renyi.hu/~p_erdos/1964-22.pdf.

Jack Edward Graver and James Yackel. Some graph theoretic results associated with Ramsey's theorem. *J. Combinatorial Theory*, 4:125–175, 1968.

Ferdinand Ihringer, Deepak Rajendraprasad, and Thilo Weinert. New bounds on the Ramsey number $r(I_m, L_n)$. *Discrete Math.*, 344(3):Paper No. 112268, 11, 2021. ISSN 0012-365X. doi:10.1016/j.disc.2020.112268. URL https://doi.org/10.1016/j.disc.2020.112268.

Jeong Han Kim. The Ramsey number $R(3, t)$ has order of magnitude $t^2 / \log t$. *Random Structures Algorithms*, 7(3): 173–207, 1995. ISSN 1042-9832. doi:10.1002/rsa.3240070302. URL https://doi.org/10.1002/rsa.3240070302.

Jean Ann Larson and William John Mitchell. On a problem of Erdős and Rado. *Ann. Comb.*, 1(3):245–252, 1997. ISSN 0218-0006. doi:10.1007/BF02558478. URL http://dx.doi.org/10.1007/BF02558478.

Stanisław P. Radziszowski. Small Ramsey numbers. *Electron. J. Combin.*, 1:Dynamic Survey 1, 30 pp. (electronic), 1994. ISSN 1077-8926. URL http://www.combinatorics.org/Surveys/index.html.

Kenneth Brooks Reid, Jr. and Ernest Tilden Parker. Disproof of a conjecture of Erdős and Moser on tournaments. *J. Combinatorial Theory*, 9:225–238, 1970.

James Bergheim Shearer. A note on the independence number of triangle-free graphs. *Discrete Math.*, 46(1):83–87, 1983. ISSN 0012-365X. doi:10.1016/0012-365X(83)90273-X.