Big Ramsey degrees of 3-uniform hypergraphs are finite

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Joint work with M. Balko, J. Hubička, M. Konečný, and L. Vena

Martin Balko, David Chodounský, Jan Hubička,
Matěj Konečný, Lluis Vena,
Big Ramsey degrees of 3-uniform hypergraphs are finite,
https://arxiv.org/abs/2008.00268

Definition

Let **A** be a countable structure. We say that **A** has *finite big Ramsey degrees* if for every $n \in \omega$ there is $D(n) \in \omega$ such that for every finite coloring of $[\mathbf{A}]^n$ there is a copy **B** of **A** (inside of **A**) such that $[\mathbf{B}]^n$ has at most $D(n)$ colors.

Example

- $(\omega,$ no structure$)$ (Ramsey)
- $(\mathbb{Q}, <)$ (Galvin, Laver, Devlin)
- Random (Rado) graph (Todorčević, Sauer)
- Triangle free Henson graph $\mathbb{H}_3$ (Dobrinen, Hubička)
- Random 3-hypergraph (BHChKV)

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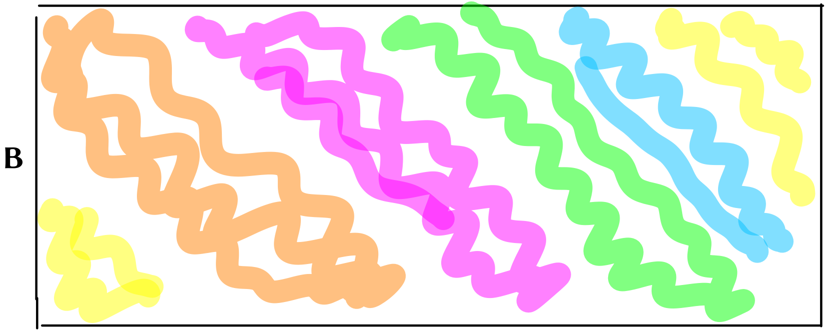

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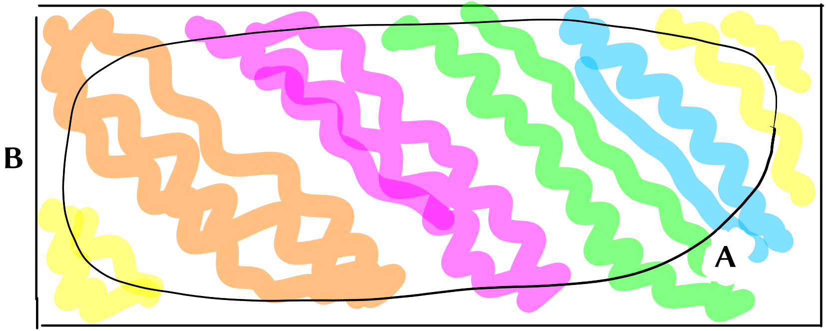

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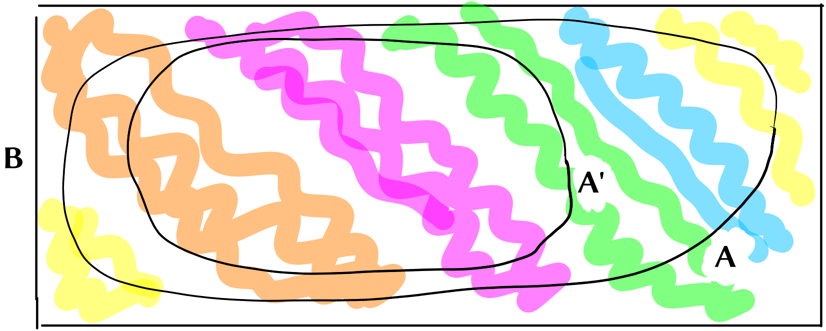

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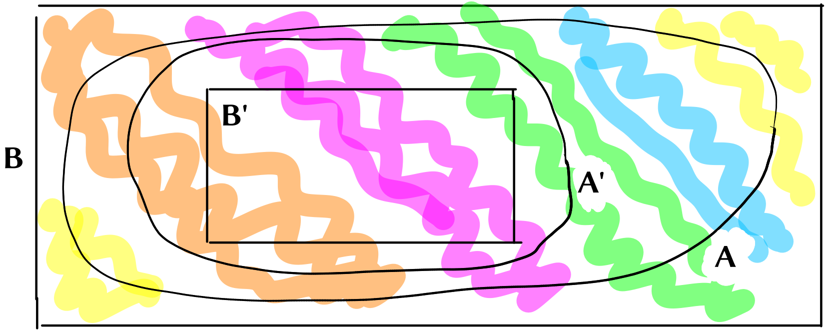

Trees

- rooted
- height at most ω ... $h(T) \leq \omega$
- finitely branching
- balanced (no short branches)
- n-th level of T ... $T(n)$
- initial subtree ... $T(<n)$
- set of immediate successors of s in T ... $isu_T(s)$

Definition

A subtree S of T of height $h(S) \in \omega + 1$ is a *strong subtree* if

- $\forall n < h(S)\ \exists m < h(T)$ such that $S(n) \subseteq T(m)$,
- $\forall s \in S\ \forall t \in isu_T(s)\ \exists!(s' \in S, s' \geq t, s' \in isu_S(s))$, unless $isu_S(s) = \emptyset$.

We write $S \in STR_n(T)$.

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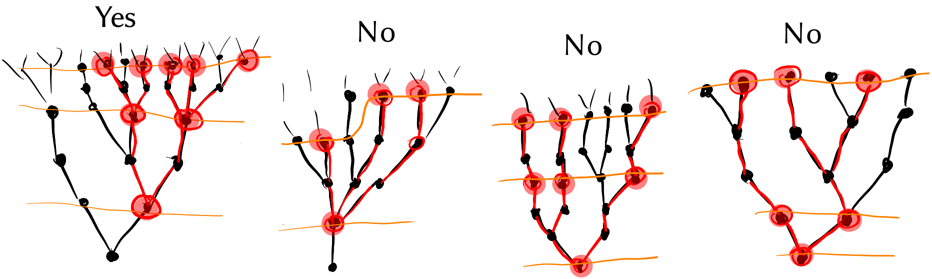

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If $S \in STR_n(T)$ and $R \in STR_m(S)$, then $R \in STR_m(T)$.

Theorem (Milliken, simple version)

If T is a tree of height ω, $n, k \in \omega$, and $\chi\colon\ STR_n(T) \to k$ is a finite coloring, then there exists $S \in STR_\omega(T)$ such that χ is monochromatic on $STR_n(S)$.

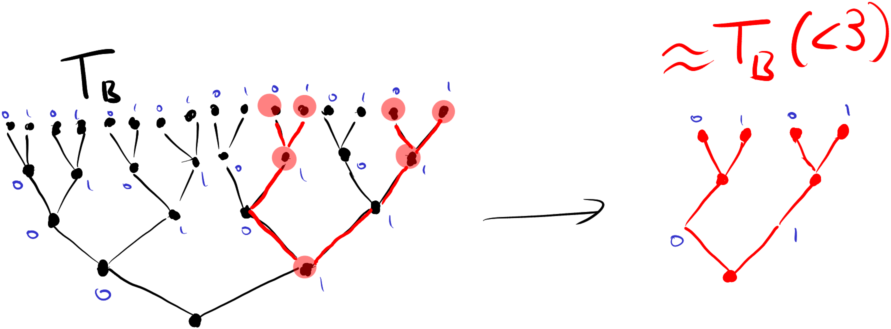
Trees, examples

Example

$\mathbf{T}_B = 2^{<\omega}$, the binary tree

Observation

If $S \in STR_n(\mathbf{T}_B)$, then S is isomorphic to $\mathbf{T}_B(<n)$.



Trees, examples

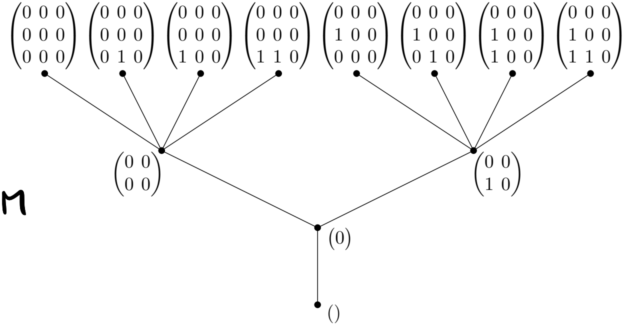
Example

$\mathbf{T}_M = \bigcup\{\, 2^{n\times n} : n \in \omega \,\}$, ordered by extension. The tree of matrices.

$\mathbf{T}_{SM} \subset \mathbf{T}_M$, the tree of sub-diagonal matrices.
If $A \in \mathbf{T}_{SM}$ and $A(i,j) \neq 0$, then $i < j$.

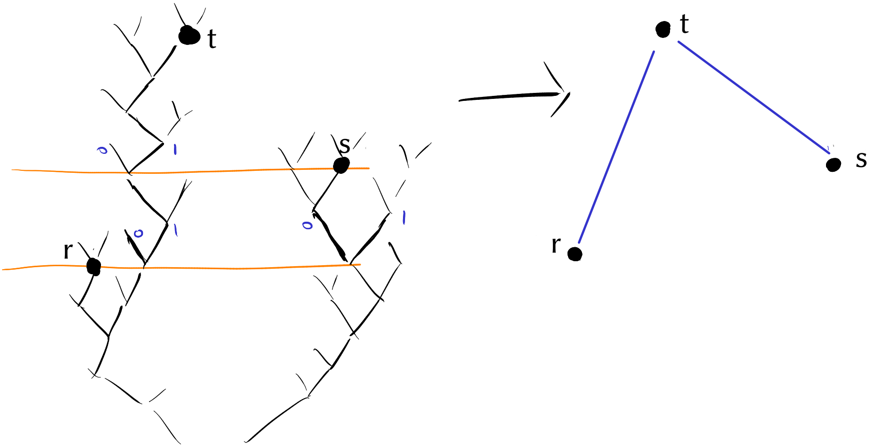
For $A \in \mathbf{T}_M(n)$ we write $|A| = n$.

T_{SM}



Random graph has finite big Ramsey degrees

For $s, t \in \mathbf{T}_B$ define $E(s, t)$ if $|s| < |t|$ and $t(|s|) = 1$.



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(both as a graph and as a tree).

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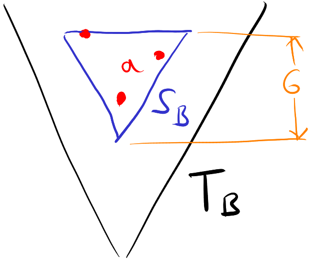
For every $n \in \omega$ and $a \in [\mathbf{T}_B]^n$ there exists
$S_B \in STR_{2n}(\mathbf{T}_B)$ such that $a \subset S_B$.



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S_B has size $2^{2n} - 1$. I.e. $[S_B]^n$ has size $< (2^{2n})^n$.

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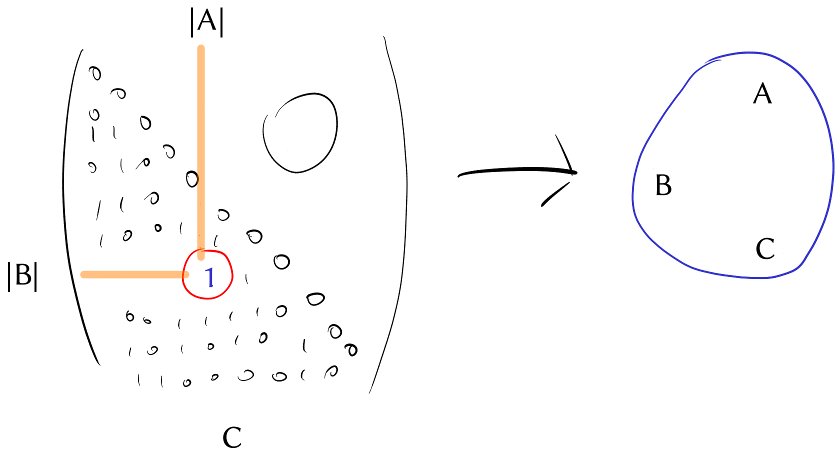
Proof

Given finite coloring $\chi \colon [\mathbf{T}_B]^n \to k$.
Induces finite coloring $\bar{\chi} \colon STR_{2n}(\mathbf{T}_B) \to k^{(2^{2n})^n}$.
Use Milliken's theorem to find $S \in STR_\omega(\mathbf{T}_B)$,
a $\bar{\chi}$-monochromatic copy of $\mathbf{T}_B$.

Universal 3-hypergraphs have finite big Ramsey degrees

For $A, B, C \in \mathbf{T}_{SM}$ define $E(A, B, C)$ is $|A| < |B| < |C|$ and $C(|A|, |B|) = 1$.



Universal 3-hypergraphs have finite big Ramsey degrees

For $A, B, C \in \mathbf{T}_{SM}$ define $E(A, B, C)$ is $|A| < |B| < |C|$ and $C(|A|, |B|) = 1$.

Proposition

The hypergraph $(\mathbf{T}_{SM}, E)$ is universal (for countable 3-hypergraphs).

Observation

If $S \in STR_\omega(\mathbf{T}_{SM})$, then (S, E) is not a copy of $(\mathbf{T}_{SM}, E)$.
(It is wider and we can find a copy of $\mathbf{T}_{SM}$ inside S.)

Problem

For $S \in STR_{2n}(\mathbf{T}_{SM})$ there is no bound on the size of S.
I.e. a finite coloring $\chi\colon [\mathbf{T}_{SM}]^n \to k$ does not induce
a finite coloring $\bar{\chi}$ of $STR_{2n}(\mathbf{T}_{SM})$.

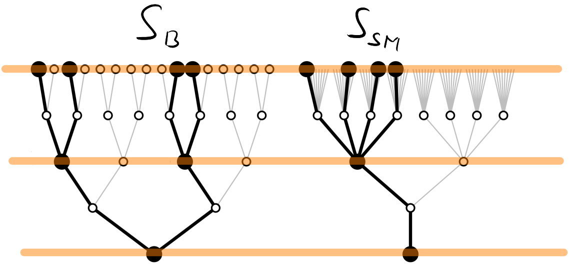
Product trees

$\mathbf{T}_{SM} \otimes \mathbf{T}_B$ ... the product tree

Definition

We say that $S_{SM} \otimes S_B \in STR_k(\mathbf{T}_{SM} \otimes \mathbf{T}_B)$ ($S_{SM} \otimes S_B$ is a strong subtree of $\mathbf{T}_{SM} \otimes \mathbf{T}_B$) if

- $S_{SM} \in STR_k(\mathbf{T}_{SM})$,
- $S_B \in STR_k(\mathbf{T}_B)$, and
- $\forall n \in k \, \exists m \in \omega$ such that $S_{SM}(n) \subseteq \mathbf{T}_{SM}(m)$ and $S_B(n) \subseteq \mathbf{T}_B(m)$.



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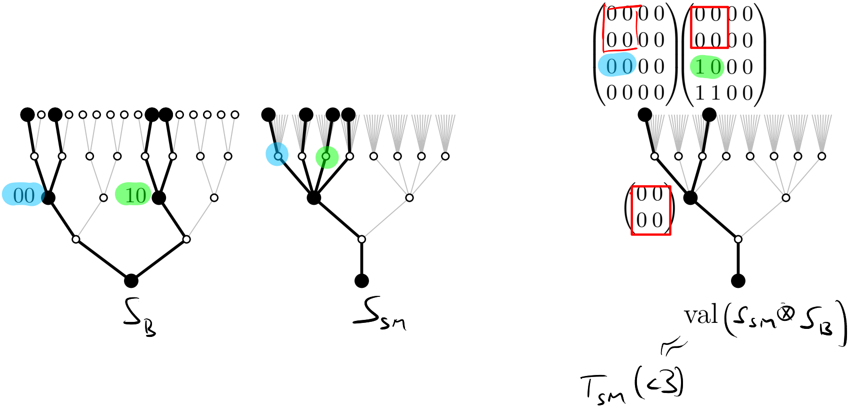
Theorem (Milliken, special case)

If $n, k \in \omega$ and $\chi : STR_n(\mathbf{T}_{SM} \otimes \mathbf{T}_B) \to k$ is a finite coloring, then there exists $S_{SM} \otimes S_B \in STR_\omega(\mathbf{T}_{SM} \otimes \mathbf{T}_B)$ such that χ is monochromatic on $STR_n(S_{SM} \otimes S_B)$.

Valuations

Suppose $S_{SM} \otimes S_B \in STR_k(\mathbf{T}_{SM} \otimes \mathbf{T}_B)$ for some $k \in \omega + 1$.
We define the tree $val(S_{SM} \otimes S_B) \subseteq S_{SM}$ by induction:

- The root of $val(S_{SM} \otimes S_B)$ is the root of S_{SM}.
- If $A \in val(S_{SM} \otimes S_B)$, $t \in S_B(|A|)$, $C \in isu_{S_{SM}}(A)$, and $C > A^\frown t$, then $C \in val(S_{SM} \otimes S_B)$.



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Observation

If $S_{SM} \otimes S_B \in STR_k(\mathbf{T}_{SM} \otimes \mathbf{T}_B)$,
then $(val(S_{SM} \otimes S_B), E)$ is a copy of $(\mathbf{T}_{SM}(< k), E)$
(both as a hypergraph and as a tree).

Lemma (false but fixable)

For every $n \in \omega$ and $a \in [\mathbf{T}_{SM}]^n$ there exists
$S_{SM} \otimes S_B \in STR_{2n}(\mathbf{T}_{SM} \otimes \mathbf{T}_B)$ such that $a \subset val(S_{SM} \otimes S_B)$.

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Proof

Given finite coloring $\chi\colon [\mathbf{T}_{SM}]^n \to k$.
Induces finite coloring $\bar{\chi}\colon\ STR_N(\mathbf{T}_{SM} \otimes \mathbf{T}_B) \to K$
(look at colors on valuations).
Use Milliken's theorem.