Tame fields and beyond, V

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Plan for this lecture

In this lecture, I will introduce extremal fields and present a result that will lead us back to deeply ramified and perfectoid fields. As a preparation, we will introduce and discuss large fields, and take the occasion to give examples for the wall of imperfection.

Take an extension $L|K$. By a place of $L|K$ we mean a place of L whose restriction to K is the identity. Note that if P is trivial on K, i.e., its restriction to K is an isomorphism σ on K, then it is equivalent to the place P' of L which is defined by $aP' := \tau^{-1}(aP)$, where τ is an extension of σ to the residue field LP, which yields that the restriction of P' to K is the identity. A place of an extension L of K is rational if $LP = K$. By what we just said, we can always assume, modulo equivalence, that it is a place of $L|K$.

Large fields

Following Florian Pop [P 1995, 1996], a field K is called a large field (or ample field) if it satisfies one of the following equivalent conditions:

(LF) *For every smooth curve over K the set of rational points is infinite if it is non-empty.*

(LF′) *In every smooth, integral variety over K the set of rational points is Zariski-dense if it is non-empty.*

(LF″) *For every function field $F|K$ in one variable the set of rational places is infinite if it is non-empty.*

For the equivalence of (LF) and (LF′), note that the set of all smooth K-curves through a given smooth K-rational point of an integral K-variety X is Zariski-dense in X. If (LF) holds, then the set of K-rational points of any such curve is Zariski-dense in the curve, which implies that the set of K-rational points of X is Zariski-dense in X.

Large fields

The equivalence of (LF) and (LF$''$) follows from two well-known facts:
a) every function field in one variable is the function field of a smooth curve (cf. [H 1977], Chap. I, Theorem 6.9), and
b) every K-rational point of a smooth curve gives rise to a K-rational place.
The latter is a special case of a much more general result:

Theorem (*)

Assume that the affine irreducible variety V defined over K has a simple K-rational point. Then its function field admits a rational place of maximal rank, centered at this point.

This follows from results in [Ab 1956] (see appendix A of [JR 1980]). For a rational place P of a function field $F|K$, to be of maximal rank means that P is the composition of $\mathrm{trdeg}\, F|K$ many places with archimedean value groups.

Large fields

Theorem (**)

The following conditions are equivalent:

1) K is a large field,

2) K is existentially closed in every function field F in one variable over K which admits a K-rational place,

3) K is existentially closed in the henselization $K(t)^h$ of the rational function field $K(t)$ with respect to the t-adic valuation,

4) K is existentially closed in the field $K((t))$,

5) K is existentially closed in every extension field which admits a discrete K-rational place.

The canonical t-adic place of the fields $K(t)^h$ and $K((t))$ is discrete, and it is trivial on K and K-rational. Therefore, 5) implies 3) and 4).

Large fields

In [L 1954], Serge Lang proved that every field K complete under a rank one valuation is large. But this already follows from the fact that such a field is henselian. Indeed, if a field K admits a non-trivial henselian valuation, then the Implicit Function Theorem holds in K (cf. [PrZi 1978]). Using this fact, it is easy to show that K satisfies (LF). On the other hand, it is also easy to prove, via an embedding lemma, that such K satisfies condition 3) of the foregoing theorem. We note:

Proposition

If a field K admits a non-trivial henselian valuation, then it is large.

Rational place = existentially closed?

In view of condition 5) of Theorem (**), the question arises whether the existence of a K-rational place of an extension field L of a large field K *always* implies that K is existentially closed in L. We will see that this is at least true for perfect fields K.

Theorem (**) leads us to ask whether large fields satisfy assertions even stronger than those in that theorem. The following has been proved in [K 2004b]:

Theorem (***)

Let K be a perfect field. Then the following conditions are equivalent:

1) K is a large field,

2) K is existentially closed in every power series field $K((\Gamma))$.

3) K is existentially closed in every extension field L which admits a K-rational place.

In particular, we obtain:

Theorem

Let K be a perfect field which admits a henselian valuation. Assume that the extension field L of K admits a K-rational place. Then K is existentially closed in L.

A field K is existentially closed in an extension field L if it is existentially closed in every finitely generated subextension F in L. If L admits a K-rational place P, then every such function field F admits a K-rational place, namely, the restriction of P. Hence, condition 3) of the foregoing theorem is equivalent to the following condition on K:

(RP=EC) *If an algebraic function field $F|K$ admits a rational place, then K is existentially closed in F.*

By Theorem (**), every field K which satisfies (RP=EC) is large. Let us see what we can say about the converse.

Places admitting smooth local uniformization

An Abhyankar place is a place whose associated valuation is an Abhyankar valuation. In [KnK 2005] the following is shown:

Theorem

Every function field with a rational discrete or rational Abhyankar place admits smooth local uniformization.

In [K 2004b] this result is extended to:

Theorem (†)

The rational discrete places, the rational places of maximal rank, and the rational Abhyankar places lie dense in the space of all rational places of $F|K$ which admit smooth local uniformization.

Note that every place of maximal rank is an Abhyankar place.

Density in Zariski spaces

Take a function field $F|K$. For a place P on F, we denote by $\mathcal{O}_P$ its valuation ring, and by $\mathcal{M}_P$ the maximal ideal.

By $S(F|K)$ we denote the set of all places of $F|K$. It is called the Zariski space (or Zariski–Riemann manifold) of $F|K$. $S(F|K)$ carries the Zariski-topology, for which the basic open sets are the sets of the form

$$\{P \in S(F|K) \mid a_1, \ldots, a_k \in \mathcal{O}_P\}, \tag{1}$$

where $k \in \mathbb{N} \cup \{0\}$ and $a_1, \ldots, a_k \in F$.

With this topology, $S(F|K)$ is a spectral space (cf. [Ho 1969]); in particular, it is quasi-compact.

Density in Zariski spaces

The density statement in Theorem (†) refers to the associated patch topology (or constructible topology), which is the finer topology whose basic open sets are the sets

$$\{P \in S(F|K) \mid a_1 \,, \ldots, \, a_k \in \mathcal{O}_P \,;\, b_1 \,, \ldots, \, b_\ell \in \mathcal{M}_P\} \,, \qquad (2)$$

where $k, \ell \in \mathbb{N} \cup \{0\}$ and $a_1, \ldots, a_k, b_1, \ldots, b_\ell \in F$. With the patch topology, $S(F|K)$ is a totally disconnected compact Hausdorff space.

The density of several special sets of places in $S(F|K)$ has been shown in [KPr 1984], [K 2004b], and [BKK 2022], using the model theory of henselian fields with residue characteristic 0 and of tame fields, and several applications have been given.

Rational place = existentially closed?

Take a function field $F|K$ with a rational place P which admits local uniformization. That is, $F|K$ admits a model on which P is centered at a simple K-rational point. By Theorem (*), F also admits a K-rational place Q of maximal rank. Hence by Theorem (†), $F|K$ also admits a rational discrete place. If K is large, then it follows from Theorem (**) that K is existentially closed in F. This proves the following well known result:

Theorem (††)

Let K be a large field and $F|K$ an algebraic function field. If there is a rational place of $F|K$ which admits local uniformization, then K is existentially closed in F.

Rational place = existentially closed?

As an immediate consequence, we obtain:

Theorem

Assume that all rational places of arbitrary function fields admit local uniformization. Then every large field satisfies (RP=EC), and the three conditions of Theorem (***) are equivalent, for arbitrary fields K.

Does the converse also hold?

Rational place = existentially closed?

Theorem (††) together with Theorem (†) implies:

Corollary

Let K be a large field and $F|K$ an algebraic function field. If there is a rational discrete or a rational Abhyankar place of $F|K$, then K is existentially closed in F.

For the case of $F|K$ admitting a rational discrete place P, the assertion is already contained in Theorem (**).
To conclude with, let us state a converse of our above results.

Theorem ([K 2004b])

Let $F|K$ be an algebraic function field such that K is existentially closed in F. Take any elements $z_1, \ldots, z_n \in F$. Then there are infinitely many (nonequivalent) rational places of $F|K$ of maximal rank which are finite on $z_1, \ldots, z_n$.

Rational place = existentially closed?

An even stronger version of this theorem has been proven and put to work in [BKK 2022]:

Theorem

Take an algebraic function field $F|K$ such that K is existentially closed in F. Take any nonzero elements $z_1, \ldots, z_n \in F$. Further, choose $r \in \mathbb{N}$ such that $1 \leq r \leq s = \operatorname{trdeg} F|K$, and an arbitrary ordering on $\mathbb{Z}^r$; denote by Γ the so obtained ordered abelian group. Then there are infinitely many (nonequivalent) rational places $P \in S(F|K)$ such that $v_P F = \Gamma$ and $z_i P \neq 0, \infty$ for $1 \leq i \leq n$.

Extremal fields

A valued field (K, v) with valuation ring $\mathcal{O}$ and value group vK is called extremal if for every multi-variable polynomial $f(X_1, \ldots, X_n)$ over K the set

$$\{vf(a_1, \ldots, a_n) \mid a_1, \ldots, a_n \in \mathcal{O}\} \subseteq vK \cup \{\infty\}$$

has a maximal element. This notion was introduced by Yuri Ershov in [Er 2004] in connection with valued skew fields which are finite-dimensional over their center. It turns out that the original definition given in that paper (and also in talks given by its author) which has "K" in place of "$\mathcal{O}$", is flawed in the sense that there are no extremal valued fields except for algebraically closed valued fields. Consequently, Proposition 2 of the cited paper, under Ershov's original definition, is false.

Ershov's "theorem"

Proposition 2 of [Er 2004] says:

Proposition

Every henselian defectless discretely valued field is extremal.

This implies that for each field k, the Laurent series field $(k((t)), v_t)$ is extremal.

Starchenko's counterexample

In 2009 during a program at the Fields Institute, I worked with Salih Azgin (now Durhan) on a classification of extremal fields. There, we met Sergei Starchenko, who provided us with the following counterexample to Ershov's proposition. Consider the polynomial

$$f(X,Y) \;=\; X^2 + (XY-1)^2$$

over the Laurent series field $(\mathbb{R}((t)), v_t)$. Observe that

$$v_t f(t^n, t^{-n}) \;=\; 2n \quad \text{for all } n \in \mathbb{N}\,,$$

but $f(a,b) \neq 0$ for all a,b in the formally real field $\mathbb{R}((t))$. Hence, the set $\{vf(a,b) \mid a,b \in \mathbb{R}((t))\}$ has no maximal element in $\mathbb{Z} \cup \{\infty\}$, which shows that Ershov's proposition does not hold with his original definition.

Proposition holds with corrected definition

However, in [AKP 2012] we were able to prove that the proposition holds with the corrected definition that we have given above. (Note that in Starchenko's example, the values $v_t t^{-n}$ are not bounded from below.)

We obtain that $(\mathbb{F}_p((t)), v_t)$ is extremal. On the other hand, it is easily seen that this property is elementary. Thus, in view of the open questions about $\mathbb{F}_p((t))$, it is important to study the algebra and model theory of extremal fields.

Elementary properties of $\mathbb{F}_p((t))$

In [K 2001], axioms about additive polynomials were formulated in the hope that adding them to the "naive axiom system" for $\mathbb{F}_p((t))$ will result in a complete system (cf. Lecture IV). Let us have a closer look. A subset A of a valued field (K,v) has the optimal approximation property if for every $z \in K$ there is some $y \in A$ such that

$$v(z-y) \;=\; \max\{v(z-x) \mid x \in A\}\,.$$

In general, y is not unique; this is why we do not talk of "best approximation". The axioms mentioned above for a valued field (K,v) of positive characteristic say that under certain conditions on an additive polynomial f in several variables, the image of K under f has the optimal approximation property. It was shown in [K 2001] that these axioms hold in all maximal fields.

It was then shown in [DK 2002] that in $\mathbb{F}_p((t))$ the conditions on the additive polynomials are not needed:

Theorem

If f is an additive polynomial in several variables with coefficients in $\mathbb{F}_p((t))$, then the image of $\mathbb{F}_p((t))$ under f has the optimal approximation property.

The proof uses that $(\mathbb{F}_p((t)), v_t)$ is locally compact.

Connection with extremality

In order to exhibit the connection between optimal approximation property and extremality, we introduce the definition of extremality with respect to specific sets of polynomials. Take a valued field (K, v) and a subset $S \subseteq K$. If f is a polynomial in n variables with coefficients in K, then we will say that (K, v) is S-extremal with respect to f in K if the set

$$vf(S^n) := \{vf(a_1, \ldots, a_n) \mid a_1, \ldots, a_n \in S\} \subseteq vK \cup \{\infty\} \tag{3}$$

has a maximum in $vK \cup \{\infty\}$. Hence (K, v) is extremal if it is $\mathcal{O}$-extremal with respect to every polynomial f in several variables with coefficients in K.

Connection with extremality

Take a field K of characteristic $p > 0$. A polynomial $h \in K[X_1, \ldots, X_n]$ is called a p-polynomial if it is of the form $f + c$, where $f \in K[X_1, \ldots, X_n]$ is an additive polynomial and $c \in K$. The proof of the following observation is straightforward:

Lemma

The images of all additive polynomials over (K, v) have the optimal approximation property if and only if K is K-extremal with respect to all p-polynomials with coefficients in K.

Connection with extremality

From this lemma, we derive in [AnK 2016]:

Theorem

If (K, v) is an extremal field of characteristic $p > 0$ with $[K : K^p] < \infty$, then the images of all additive polynomials have the optimal approximation property.

Open problem: Does the assertion of this theorem fail in the case of $[K : K^p] = \infty$?

An axiom system for $\mathbb{F}_p((t))$

We see that the following axiom system is satisfied by $\mathbb{F}_p((t))$:

$$\left.\begin{array}{l}\text{“henselian defectless extremal field}\\ \text{of characteristic } p\\ \text{whose value group is a } \mathbb{Z}\text{-group,}\\ \text{and whose residue field is } \mathbb{F}_p\text{”.}\end{array}\right\} \tag{4}$$

On the other hand, we know that the assertion that the image of all additive polynomials has the optimal approximation property follows from these axioms. Therefore, this axiom system is a good (and elegant) candidate for the axiomatization of $\mathbb{F}_p((t))$. Unfortunately, it is still not known whether it is complete. We need to know more about extremal fields.

Applications of the notion of extremality

Before I turn to the structure theory of extremal fields, let me mention that the flexible notion of extremality also enables us to show that certain properties of valued fields are elementary. For example,

• (K, v) is algebraically maximal if and only if it is K-extremal (or $\mathcal{O}$-extremal) with respect to all polynomials in one variable with coefficients in K.

• (K, v) is inseparably defectless if and only if it is K-extremal (or $\mathcal{O}$-extremal) with respect to certain p-polynomials.

For more details, see [K 2010a].

Characterization of extremal fields

In 2009 at the Fields Institute, we co-opted Florian Pop, as an expert for large fields, for our project of characterizing extremal fields. In [AKP 2012], we gave a partial characterization. This was improved in joint work with Sylvy Anscombe in [AnK 2016], but it is still partial:

Theorem

Let (K, v) be a nontrivially valued field. If (K, v) is extremal, then it is henselian defectless and

(i) *vK is a $\mathbb{Z}$-group, or*

(ii) *vK is divisible and Kv is large.*

Conversely, if (K, v) is henselian defectless and

(i) *$vK \simeq \mathbb{Z}$, or vK is a $\mathbb{Z}$-group and $\mathrm{char}\, Kv = 0$, or*

(ii) *vK is divisible and Kv is large and perfect,*

then (K, v) is extremal.

Maximality does not imply extremality

After I was introduced to Ershov's notion of extremality, for some time I guessed that all maximal valued fields should be extremal. However, the conditions on the value groups and residue fields clearly show that this is not the case. Conversely, extremal fields need not be maximal (which is clear, as the former property is elementary, while the latter is not.)

Characterization of extremal fields

Things become easy for tame fields. Tame fields of positive residue characteristic $p > 0$ are henselian defectless, and they have p-divisible value groups which consequently are not $\mathbb{Z}$-groups. On the other hand, all henselian defectless valued fields with divisible value group and perfect residue field are tame fields. Therefore, in the case of positive residue characteristic and value groups that are not $\mathbb{Z}$-groups, the theorem is in fact talking about tame fields:

Theorem

A tame field of positive residue characteristic is extremal if and only if its value group is divisible and its residue field is large.

Extremality is stronger than optimal approximation property

Corollary

There are perfect non-extremal fields of positive characteristic in which the images of all additive polynomials have the optimal approximation property.

This shows that extremality is stronger than the assertion that the images of all additive polynomials have the optimal approximation property. This begs the questions:

Are there elementary properties of additive polynomials that are not expressed by the optimal approximation property?

Or are there other elementary properties of extremal fields of positive characteristic that do not follow from those connected with additive polynomials and the “naive axiom system”?

Characterization of extremal fields

Beyond tame fields, we hit the wall of imperfection. The gap in the above characterization is made obvious by the following:

Proposition

a) There are henselian defectless valued fields (K, v) of positive characteristic with value group a $\mathbb{Z}$-group that are extremal, and others that are not.

b) There are henselian defectless valued fields (K, v) of mixed characteristic with value group a $\mathbb{Z}$-group that are extremal, and others that are not.

c) There are henselian defectless nontrivially valued fields (K, v) of positive characteristic with divisible value group and imperfect large residue field that are extremal, and others that are not.

d) There are henselian defectless valued fields (K, v) of mixed characteristic with divisible value group and imperfect large residue field that are extremal, and others that are not.

Characterization of extremal fields

None of the non-extremal fields that we construct for the proof of this proposition are maximal. This leads us to the following

Conjecture: Every maximal field with value group a $\mathbb{Z}$-group, or divisible value group and large residue field, is extremal.

A theorem on $\aleph_1$-saturated valued fields

The following theorem provides a compelling way of constructing maximal extremal fields and is used in the proof of parts c) and d) of the previous theorem.

Theorem

Let (K, v) be any $\aleph_1$-saturated valued field. Assume that Γ and Δ are convex subgroups of vK such that $\Delta \subsetneq \Gamma$ and Γ/Δ is archimedean. Let u (respectively w) be the coarsening of v corresponding to Δ (resp. Γ). Denote by $\bar{u}$ the valuation induced on Kw by u. Then $(Kw, \bar{u})$ is maximal, extremal and large, and its value group is isomorphic either to $\mathbb{Z}$ or to $\mathbb{R}$. In the latter case, also $Ku = (Kw)\bar{u}$ is large.

Application to deeply ramified fields

Every perfectoid field is a henselian semitame field of rank 1. Take an $\aleph_1$-saturated elementary extension (K, v) of such a field. Then it is also henselian and semitame.

Assume that (K, v) is of mixed characteristic. Consider the canonical decomposition $v = v_0 \circ v_p \circ \bar{v}$ which we introduced in Lecture IV. By the previous theorem, (Kv_0, v_p) is maximal, hence defectless. As (K, v_0) has residue characteristic 0, it is also defectless. Consequently, $(K, v_0 \circ v_p)$ is defectless, and as $v_0 \circ v_p$ is a coarsening of the henselian valuation v, it is also henselian.

On the other hand, the following fact was shown in [KR 2022]:

• If (K, v) is a semitame field, then the same holds for every coarsening of v.

Hence we find that $(K, v_0 \circ v_p)$ is a henselian defectless semitame field, and therefore a tame field.

Further, as vK is an elementary extension of a p-divisible and thus regular ordered abelian group, and as the value group of $(K, v_0 \circ v_p)$ is the quotient of vK with respect to the nontrivial convex subgroup $\bar{v}(Kv_0 \circ v_p)$, we see that the value group of $(K, v_0 \circ v_p)$ is divisible.