Tame fields and beyond, III

Franz-Viktor Kuhlmann
University of Szczecin, Poland

Lecture series, Vienna, November and December 2022

Take a valued field (K, v) and set $p = \operatorname{char} Kv$ if this is positive, and $p = 1$ otherwise. We denote by $\tilde{K}$ the algebraic and by K^{sep} the separable-algebraic closure of K. Further, we choose an arbitrary extension $\tilde{v}$ of v to $\tilde{K}$.
Take an algebraic extension $L|K$. Then for every $\sigma \in \operatorname{Aut} \tilde{K}|K$, the map

$$\tilde{v}\sigma \;=\; \tilde{v} \circ \sigma : \; L \ni a \;\mapsto\; \tilde{v}(\sigma a) \in \tilde{v}\tilde{K}$$

is a valuation of L which extends v.

Theorem

The set of all extensions of v from K to L is

$$\{\tilde{v}\sigma \mid \sigma \text{ an embedding of } L \text{ in } \tilde{K} \text{ over } K\} \,.$$

(We say that "all extensions of v from K to L are conjugate".)

Ramification groups

From now on, we assume that the algebraic extension $L|K$ is normal. Hence the set of all extensions of v from K to L is given by $\{\tilde{v}\sigma \mid \sigma \in \operatorname{Aut} L|K\}$. For simplicity, we denote the restriction of $\tilde{v}$ to L again by v. The valuation ring of v on L will be denoted by $\mathcal{O}_L$, and its unique maximal ideal by $\mathcal{M}_L$. We will now define distinguished subgroups of $G := \operatorname{Aut} L|K$. The subgroup

$$\begin{aligned} G^d &= G^d(L|K,v) := \{\sigma \in G \mid \forall x \in \mathcal{O}_L : v\sigma x \geq 0\} \\ &= \{\sigma \in G \mid \sigma\mathcal{O}_L \subseteq \mathcal{O}_L\} \end{aligned}$$

is called the decomposition group **of** $(L|K,v)$. It is easy to show that σ sends $\mathcal{O}_L$ into itself if and only if the valuations v and $v\sigma$ agree on L. Thus,

$$G^d = \{\sigma \in G \mid v\sigma = v \text{ on } L\}\,.$$

Further, the inertia group is defined to be

$$G^i = G^i(L|K,v) := \{\sigma \in G \mid \forall x \in \mathcal{O}_L : v(\sigma x - x) > 0\}\,,$$

and the ramification group is

$$G^r = G^r(L|K,v) := \{\sigma \in G \mid \forall x \in L^\times : v(\sigma x - x) > vx\}\,.$$

Ramification fields

The fixed fields of G^d, G^i and G^r in the maximal separable extension field L_s of K within L are called decomposition field, inertia field and ramification field of $(L|K,v)$, respectively. For simplicity, let us abbreviate them by Z, T and V, respectively. (These letters refer to the German words “Zerlegungskörper”, “Trägheitskörper” and “Verzweigungskörper”.)

Remark: In contrast to the classical definition used by other authors, we take decomposition field, inertia field and ramification field to be the fixed fields of the respective groups *in the maximal separable subextension*. The reason for this will become clear later.

By our definition, V, T and Z are separable-algebraic extensions of K, and $L_s|\mathrm{V}$, $L_s|\mathrm{T}$, $L_s|\mathrm{Z}$ are (not necessarily finite) Galois extensions. Further,

$$1 \subset G^r \subset G^i \subset G^d \subset G \text{ and thus, } L_s \supset \mathrm{V} \supset \mathrm{T} \supset \mathrm{Z} \supset K\,.$$

Theorem

G^i and G^r are normal subgroups of G^d, and G^r is a normal subgroup of G^i. Therefore, $\mathrm{T}|\mathrm{Z}$, $\mathrm{V}|\mathrm{Z}$ and $\mathrm{V}|\mathrm{T}$ are (not necessarily finite) Galois extensions.

The decomposition field

First, we consider the decomposition field Z. In some sense, it represents all extensions of v from K to L.

Theorem

a) $v\sigma = v\tau$ *on* L *if and only if* $\sigma\tau^{-1}$ *is trivial on* Z.
b) $v\sigma = v$ *on* Z *if and only if* σ *is trivial on* Z.
c) The extension of v *from* Z *to* L *is unique.*
d) The extension $(Z|K, v)$ *is immediate.*

Assertions a), b) and c) are easy consequences of the definition of G^d. For assertion d), there is a simple proof using a trick which is mentioned in the very useful appendix of the paper [A 1971] by James Ax.

The inertia field

Now we turn our attention to the inertia field T. For every $\sigma \in G^d(L|K,v)$ we have that $\sigma\mathcal{O}_L = \mathcal{O}_L$, and it follows that $\sigma\mathcal{M}_L = \mathcal{M}_L$. Hence, every such σ induces an automorphism $\overline{\sigma}$ of $\mathcal{O}_L/\mathcal{M}_L = Lv$ which satisfies $\overline{\sigma}\,\overline{a} = \overline{\sigma a}$. Since σ fixes K, it follows that $\overline{\sigma}$ fixes Kv.

Lemma

Since $L|K$ is normal, the same is true for $Lv|Kv$. The map

$$G^d(L|K,v) \ni \sigma \;\mapsto\; \overline{\sigma} \in \mathrm{Aut}\, Lv|Kv \tag{1}$$

is a group homomorphism.

The inertia field

Theorem

a) The homomorphism (1) is onto and induces an isomorphism

$$\mathrm{Aut}\,\mathrm{T}|\mathrm{Z} \;=\; G^d/G^i \;\simeq\; \mathrm{Aut}\,\mathrm{T}v|\mathrm{Z}v\,. \tag{2}$$

b) For every finite subextension $F|\mathrm{Z}$ *of* $\mathrm{T}|\mathrm{Z}$*,*

$$[F:\mathrm{Z}] \;=\; [Fv:\mathrm{Z}v]\,. \tag{3}$$

c) We have that $v\mathrm{T} = v\mathrm{Z} = vK$*. Further,* $\mathrm{T}v$ *is the relative separable-algebraic closure of* Kv *in* Lv*, and therefore,*

$$\mathrm{Aut}\,\mathrm{T}v|\mathrm{Z}v \;=\; \mathrm{Aut}\,Lv|Kv\,. \tag{4}$$

p'-division

We will now turn to the ramification field. We need a quick preparation.
Given any extension $\Delta \subset \Delta'$ of abelian groups, the p'-divisible closure of Δ in Δ' is defined to be the subgroup

$$\{\alpha \in \Delta' \mid \exists n \in \mathbb{N} : p \nmid n \wedge n\alpha \in \Delta\}$$

of all elements in Δ' whose order modulo Δ is prime to p.

The ramification field

Theorem

a) There is an isomorphism

$$\mathrm{Aut}\,\mathrm{V}|\mathrm{T} \;=\; G^i/G^r \;\simeq\; \mathrm{Hom}\left(v\mathrm{V}/v\mathrm{T}\,,\,(\mathrm{T}v)^\times\right)\,. \tag{5}$$

Since the character group on the right hand side is abelian, $\mathrm{V}|\mathrm{T}$ *is an abelian Galois extension.*
b) For every finite subextension $F|\mathrm{T}$ *of* $\mathrm{V}|\mathrm{T}$*,*

$$[F:\mathrm{T}] \;=\; (vF:v\mathrm{T})\,. \tag{6}$$

c) We have that $\mathrm{V}v = \mathrm{T}v$*, and* $v\mathrm{V}$ *is the* p'*-divisible closure of* vK *in* vL*.*

The ramification field

Theorem

The ramification group G^r is a p-group and therefore, $L_s|V$ is a p-extension and the degree of every finite subextension of $L|V$ is a power of p.
Further, vL/vV is a p-group, and the residue field extension $Lv|Vv$ is purely inseparable.
If $\mathrm{char}\, Kv = 0$, then $V = L_s = L$.

Absolute ramification theory

Absolute ramification theory is ramification theory applied to the normal extension $\tilde{K}|K$. We fix an extension of v to $\tilde{K}$ and denote it again by v. We denote by $\widetilde{vK}$ the divisible hull of vK.

We assume that v is non-trivial. This implies

$$v\tilde{K} = vK^{\mathrm{sep}} = \widetilde{vK},$$

and

$$\tilde{K}v = K^{\mathrm{sep}}v = \widetilde{Kv}.$$

Now we can present the basic facts of absolute ramification theory in the following picture.

Absolute ramification theory

Galois group		field			value group		residue field	
		$\tilde{K}$			$\widetilde{vK}$		$\widetilde{Kv}$	
			purely inseparable		$\Vert$		$\Vert$	
1		K^{sep}		separable-algebraic closure	$\widetilde{vK}$		$\widetilde{Kv}$	
			Galois p-extension			division by p		purely inseparable
G^r		K^r		absolute ramification field	$\frac{1}{p'^\infty}vK$		$(Kv)^{\mathrm{sep}}$	
	Char		abelian Galois p'-extension			division prime to p	$\Vert$	
G^i		K^i		absolute inertia field	vK		$(Kv)^{\mathrm{sep}}$	
	Gal Kv		Galois		$\Vert$			Galois
G^d		$K^d = K^h$		absolute decomposition field = henselization	vK		Kv	
					$\Vert$		$\Vert$	
Gal K		K			vK		Kv	

Classically, an algebraic extension $(L|K,v)$ is called unramified if $vL = vK$ and the algebraic extension $Lv|Kv$ is separable, and it is called tamely ramified if vL/vK is not divisible by the residue characteristic and the algebraic extension $Lv|Kv$ is separable. Note that even if the extension is unibranched, these properties do not imply that the extension is defectless. Therefore, they do not quite fit our purpose. The following definition fits better.

A unibranched extension $(L|K,v)$ is called a tame extension if every finite subextension $E|K$ satisfies the following conditions:

(TE1) the ramification index $(vE : vK)$ is not divisible by char Kv,

(TE2) the residue field extension $Ev|Kv$ is separable,

(TE3) the extension $(E|K,v)$ is defectless.

Hence a henselian field (K,v) is a tame field if and only if all of its algebraic extensions are tame extensions.

Let us assume that (K, v) is henselian. Then every algebraic extension $(L|K, v)$ is unramified, and it is unramified and defectless if and only if $L \subseteq K^i$, and tamely ramified and defectless if and only if $L \subseteq K^r$. The latter can also be expressed as follows:

Theorem

For a henselian field, its absolute ramification field is the maximal tame extension.

If L does not lie in K^r, then it is said to have wild ramification.

Elimination of Ramification

Ramification (whether tame or wild) is the valuation theoretical expression of the failure of the Implicit Function Theorem. This in turn expresses that a point on a variety, represented by a valuation on the function field, is not smooth. So we wish to eliminate ramification in a given valued function field $(F|K,v)$. That is, we wish to show that $(F|K,v)$ is inertially generated, meaning that it admits a transcendence basis $\mathcal{T}$ such that F lies in $K(\mathcal{T})^i$. In [KnK 2009] it was shown that if $(F|K,v)$ admits (smooth) local uniformization, then it must be inertially generated. This shows that (smooth) local uniformization implies elimination of ramification. In the previous lecture we have seen that valued function fields with Abhyankar valuations are inertially generated. We have also seen that we may not find $\mathcal{T}$ such that F lies in $K(\mathcal{T})^h$ ("is henselian generated").

Inertial generation and the embedding problem

If $(F|K,v)$ is inertially generated and $\mathcal{T}$ is a transcendence basis such that $F \subset K(\mathcal{T})^i$, then our embedding problem is reduced to finding an embedding of $K(\mathcal{T})$ in K^*. Indeed, since (K^*,v^*) is henselian, the embedding of $K(\mathcal{T})$ can be extended to an embedding of its henselization $K(\mathcal{T})^h$ in K^*. Now the extension $F.K(\mathcal{T})^h|K(\mathcal{T})^h$ is of the same degree as its residue field extension, which is just the separable extension $Fv|K(\mathcal{T})v$. Using Hensel's Lemma, the embedding of Fv in K^*v^* can be lifted to an embedding of $F.K(\mathcal{T})^h$ in K^*. The restriction of this embedding to F is the embedding we are looking for.

Inertial generation and the embedding problem

As we have seen, in the case of an Abhyankar valuation v the embeddings of vF and Fv can be lifted to an embedding of $K(\mathcal{T})$. This proves our embedding lemma in the case of Abhyankar valuations.

Inertial generation of a function field also reduces the problem of local uniformization to the rational function field $(K(\mathcal{T})|K, v)$. In the case of an Abhyankar valuation, it was shown in [KnK 2005] that also this rational function field admits smooth local uniformization, proving that all function fields with Abhyankar valuations admit smooth local uniformization.

Analysis of defect extension

We have seen that for the purpose of elimination of ramification, we have to consider defects in valued function fields. In the case of function fields with Abhyankar valuations, the Generalized Stability Theorem shows that there are no defects, provided that the base field itself is a defectless field. The proof of that theorem uses ramification theory to analyse finite extensions, reducing them to the case of normal extensions of degree $p = \mathrm{char}\, Kv$. Then normal forms for such extensions are derived, which show that they are defectless.

Analysis of defect extension

On the other hand, in the setting of the Henselian Rationality Theorem, defects will occur. Again, the problem is reduced to normal extensions of degree p. This time, the derived normal forms serve to replace the element x by a better choice in order to reduce the degree

$$[F.K(x)^h : K(x)^h] .$$

In all cases, the reduction to normal extensions of degree p must preserve defects and even more detailed information about the extensions. Let us introduce one of such details that will play an important role.

Cuts

Take a totally ordered set $(T,<)$. A subset $S \subseteq T$ is called an initial segment of T if for each $s \in S$ every $t < s$ also lies in S. Similarly, $S \subseteq T$ is called a final segment of T if for each $s \in S$ every $t > s$ also lies in S. A pair (Λ^L, Λ^R) of subsets of T is called a cut in T if Λ^L is an initial segment of T and $\Lambda^R = T \setminus \Lambda^L$; it then follows that Λ^R is a final segment of T. To compare cuts in $(T,<)$ we will use the lower cut sets comparison. That is, for two cuts $\Lambda_1 = (\Lambda_1^L, \Lambda_1^R)$ and $\Lambda_2 = (\Lambda_2^L, \Lambda_2^R)$ in T we will write $\Lambda_1 < \Lambda_2$ if $\Lambda_1^L \subsetneq \Lambda_2^L$, and $\Lambda_1 \leq \Lambda_2$ if $\Lambda_1^L \subseteq \Lambda_2^L$.

Cuts

Take a nonempty subset M of T. For an element $t \in T$ we will write $M < t$ if $s < t$ for every $s \in M$, and $M > t$ if $s > t$ for every $s \in M$. We define M^+ to be the cut (Λ^L, Λ^R) in T where Λ^L is the smallest initial segment containing M, that is,

$$M^+ \;=\; (\{t \in T \mid \exists\, m \in M \; t \leq m\}, \{t \in T \mid t > M\})\,.$$

Likewise, we denote by M^- the cut (Λ^L, Λ^R) in T where Λ^L is the largest initial segment disjoint from M, i.e.,

$$M^- \;=\; (\{t \in T \mid t < M\}, \{t \in T \mid \exists\, m \in M \; t \geq m\})\,.$$

Distances

Recall that for every extension $(K(z)|K,v)$,

$$v(z-K) \;:=\; \{v(z-c)\,|\,c\in K\}\,.$$

The set $v(z-K)\cap vK$ is an initial segment of vK and thus the lower cut set of a cut in vK. However, it is more convenient to work with the cut

$$\mathrm{dist}\,(z,K) \;:=\; (v(z-K)\cap vK)^+ \;\text{ in the divisible hull } \widetilde{vK} \text{ of } vK\,.$$

We call this cut the distance of z from K. The lower cut set of $\mathrm{dist}\,(z,K)$ is the smallest initial segment of $\widetilde{vK}$ containing $v(z-K)\cap vK$.

Note that $v(z-K)\subseteq vK$ if $(K(z)|K,v)$ is immediate.

Comparing distances

If $(L|K,v)$ is an algebraic extension, then $\widetilde{vL} = \widetilde{vK}$. Thus dist (z,K) and dist (z,L) are cuts in the same value group and we can compare these cuts by set inclusion of the lower cut sets. Since $v(z-K) \subseteq v(z-F)$,

$$\mathrm{dist}\,(z,K) \;\leq\; \mathrm{dist}\,(z,F)\,.$$

(Recall that in a valued field (K,v), two elements a and b are "close to each other" if $v(a-b)$ is *large*.)

Lifting defect extensions

The following fact is very important for the investigation of defect extensions:

Proposition

Take a tame extension $(N|K,v)$.

1) For every finite extension $(L|K,v)$,

$$d(L|K,v) \;=\; d(L.N|N,v)\,.$$

In particular, (K,v) is a defectless field if and only if (N,v) is.

2) If $(K(z)|K,v)$ is an immediate extension, then

$$\mathrm{dist}\,(z,K) \;=\; \mathrm{dist}\,(z,N)\,.$$

Lifting defect to the henselization

We can always reduce the analysis of unibranched defect extensions $(L|K,v)$ to the case of henselian (K,v):

Proposition

Take any valued field (K,v).

1) For every finite unibranched extension $(L|K,v)$,

$$[L:K] \;=\; [L.K^h : K^h] \quad \textit{and} \quad d(L|K,v) \;=\; d(L.K^h|K^h,v)\,.$$

2) If $(K(z)|K,v)$ *is an immediate unibranched extension, then*

$$\mathrm{dist}\,(z,K) \;=\; \mathrm{dist}\,(z,K^h)\,.$$

Lifting defect to the absolute ramification field

If we want to investigate a defect extension $(L|K,v)$, then we can consider the extension $(L.K^r|K^r,v)$ which has the same defect. The fact that G^r is a p-group (where $p = \mathrm{char}\, Kv$), i.e., $K^{\mathrm{sep}}|K^r$ is a p-extension, implies that $L.K^r|K^r$ is a tower of normal extensions of degree p. Now we can investigate each of these extensions in the tower separately.

If $\mathrm{char}\, K = p$, then every separable among these extensions is an Artin-Schreier extension, i.e., generated by a root ϑ of a polynomial of the form $X^p - X - a$; in this case, ϑ is called an Artin-Schreier generator of the extension. If $\mathrm{char}\, K = 0$, then every such extension is a Kummer extension, i.e., generated by a root η of a polynomial of the form $X^p - a$, since K^r contains all p-th roots of unity; in this case, η is called a Kummer generator of the extension.

References

- [A 1971] Ax, J.: A metamathematical approach to some problems in number theory. Proc. Symp. Pure Math., **20**, Amer. Math. Soc., 161–190 (1971)
- [KnK 2005] Knaf, H. – Kuhlmann, F.-V.: *Abhyankar places admit local uniformization in any characteristic*, Ann. Scient. Ec. Norm. Sup., **38** (2005), 833–846
- [KnK 2009] Knaf, H. – Kuhlmann, F.-V.: *Every place admits local uniformization in a finite extension of the function field*, Adv. Math., **221** (2009), 428–453

More detailed information

A lecture series on valued function fields and the defect can be found on the web page

https://math.usask.ca/fvk/Fvkls.html.

Preprints and further information:

The Valuation Theory Home Page
http://math.usask.ca/fvk/Valth.html.