Tame fields and beyond, II

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Lecture series, Vienna, November and December 2022

The Wall of Imperfection

When it comes to the algebra and model theory of valued fields in positive characteristic, we often run up against what I call the Wall of Imperfection. Observe that all tame fields are perfect, while $\mathbb{F}_p((t))$ is not. We will now scratch a bit on that wall.

Separably tame fields

A valued field (K, v) is called a separably defectless field if each of its finite separable extensions is defectless, and it is said to be a separably tame field if it is henselian and satisfies:

(ST1) if char $Kv = p > 0$, then vK is p-divisible,

(ST2) Kv is perfect,

(ST3) (K, v) is a separably defectless field.

Separably defectless fields of characteristic 0 are defectless fields, and separably tame fields of characteristic 0 are tame fields; so when talking about separably tame fields we will assume that they have positive characteristic.

Separably tame fields are not necessarily perfect; for example, every separable-algebraically closed non-trivially valued field is a separably tame field.

Separably tame fields

However, separably tame fields are very close to being perfect:

Theorem

Every separably tame field is dense in its perfect hull, or equivalently, its completion is perfect.

Let us sketch the proof. Take a separably tame field (K, v). The assertion is trivial if $\mathrm{char}\, K = 0$, thus we assume that $\mathrm{char}\, K = p > 0$. Then also $\mathrm{char}\, Kv = p$. As vK is p-divisible and Kv is perfect, the perfect hull of (K, v) is an immediate extension. Suppose that there is a finite subextension that does not lie in the completion. One then shows that there is also such a subextension of degree p; let η be its generator. By our choice of the extension, the set

$$v(\eta - K) \;:=\; \{v(\eta - c) \mid c \in K\}$$

is bounded in vK.

Transforming defect extensions

The minimal polynomial of η over K is $X^p - \eta^p$. One shows that if $b \in K$ with vb large enough and z is a root of

$$X^p - bX - \eta^p ,$$

then for every $c \in K$,

$$v(z-c) \;=\; v(\eta - c) \,.$$

It then follows that also the separable extension $(K(z)|K,v)$ is unibranched and immediate, and thus a defect extension, contradicting our assumption that (K,v) is a separably tame field.

Refining the construction

Instead of using b, we can put b^{p-1} to obtain the polynomial $X^p - b^{p-1}X - \eta^p$. Then we can set $Y = bX$ to obtain the polynomial $b^pY^p - b^pY - \eta^p$, then divide by b^p to obtain the Artin-Schreier polynomial

$$Y^p - Y - \frac{\eta^p}{b^p}\,.$$

Let ϑ be a root of this polynomial, then $(K(\vartheta)|K, v)$ is a Galois extension, and it is a defect extension by the previous argument. We note:

Proposition

Every nontrivial finite immediate purely inseparable extension that does not lie in the completion gives rise to a Galois defect extension of prime degree.

Dependent and independent defect

Via ramification theory, the study of separable defect extensions can be reduced to the study of Galois defect extensions of prime degree, as will be explained later. For valued fields of positive characteristic, a classification of these defect extensions was introduced in [K 2010a]:

a Galois defect extension of prime degree is called dependent if it is obtained from a purely inseparable extension in the way described above; otherwise, it is called independent.

The Artin-Schreier defect extension given in Abhyankar's example is independent; it cannot be dependent as it is an extension of a perfect field. Later on, this classification will play an important role, and it will also be extended to valued fields of mixed characteristic.

Model theory of separably tame fields

Let us quickly list some model theoretic results about separably tame fields.

Theorem ([K 2016])

Every separable extension $(L|K,v)$ of a separably tame field satisfies the $AKE^{\exists}$ Principle.

For the next theorems, we need some preparation.

The p-degree

We are presently working with fields of characteristic $p > 0$, so K^p is a subfield of K. We write $[K : K^p] = p^e$ where $e \geq 0$, with $e = \infty$ if the extension is infinite. Then e is called the p-degree (or degree of imperfection or Ershov invariant) of K.

AKE Principles for separably tame fields

The following theorem is a reformulation of results proven in [KPa 2016]:

Theorem

a) Every separable extension $(L|K,v)$ of separably tame fields (K,v) and (L,v) of equal finite p-degree satisfies the AKE$^{\prec}$ Principle.

c) Every two separably tame fields (K,v) and (L,v) of equal finite p-degree satisfy the AKE$^{\equiv}$ Principle.

Decidability for separably tame fields

As an immediate consequence of the $\mathrm{AKE}^{\equiv}$ Principle, we obtain the following criterion for decidability:

Theorem

Let (K, v) be a separably tame field of finite p-degree. Assume that the theories $\mathrm{Th}(vK)$ of its value group (as an ordered group) and $\mathrm{Th}(Kv)$ of its residue field (as a field) both admit recursive elementary axiomatizations. Then also the theory of (K, v) as a valued field admits a recursive elementary axiomatization and is decidable.

We do not know any analogous results for infinite p-degree.

The horror of infinite p-degree

The following was shown by Masuyoshi Nagata in [N 1962] (see also [K 2011]):

Theorem

There are valued fields of infinite p-degree which admit a finite maximal immediate extension.

The following result inspired by Nagata's result and proven in [BlK 2015a] arguably shows the most extreme failure of uniqueness of maximal immediate extensions:

Theorem

There are valued fields of infinite p-degree which admit both an algebraic maximal immediate extension as well as one of infinite transcendence degree.

Modification of Stability Theorem and Henselian Rationality Theorem

The proof of the model theoretic results uses the following modification of the Generalized Stability Theorem and of the Henselian Rationality Theorem.

Theorem ([K 2010])

Assume that v is an Abhyankar valuation on the function field $F|K$, not necessarily trivial on K. If vK is cofinal in vF and (K,v) is a separably defectless field, then (F,v) is a separably defectless field.

Theorem ([K 2019])

Let (K,v) be a separably tame field and (F,v) an immediate function field over (K,v). Assume that $F|K$ is a separable extension of transcendence degree 1. Then there is $x \in F$ such that $F \subset K(x)^h$.

Henselian rationality and defect

The reason why the Henselian Rationality Theorem is hard to prove in positive characteristic is because we have to work around the defect. Take an immediate function field $(F|K,v)$ of transcendence degree 1 and pick any element $x \in F$ transcendental over K.

Since (F,v) is an immediate extension of (K,v), it is an immediate algebraic extension of $(K(x),v)$. Observe that $(K(x),v)^h$ is an immediate extension of $(K(x),v)$ and $(F,v)^h$ is an immediate extension of (F,v).

By a result from ramification theory, F^h is the compositum of F and $K(x)^h$. Being an immediate extension is transitive, hence $(F,v)^h$ is an immediate extension of $(K(x),v)$. Consequently, $(F,v)^h$ is an immediate extension of $(K(x),v)^h$.

Henselian rationality and defect

If this extension is trivial, then we have proved that $F \subset K(x)^h$. If it is not, then we have nontrivial defect: since

$$(vF^h : vK(x)^h) = 1 \quad \text{and} \quad [F^h v : K(x)^h v] = 1\,,$$

the defect of the extension is equal to its degree. Unfortunately, this case can always happen in positive characteristic, indicating that we have chosen the wrong transcendental element x. The method of proof of the Henselian Rationality Theorem is to decrease the degree $[F^h : K(x)^h]$ step by step until we reach degree 1.

A tool for proving AKE Principles

Using basic model theoretic principles (such as Robinson's Test), the proof of AKE$^{\equiv}$ Principles can be reduced to the proof of AKE$^{\prec}$ Principles, and that in turn can be reduced to the proof of AKE$^{\exists}$ Principles. The latter can be reformulated as an algebraic problem: we have to prove embedding lemmas of the following type.

Reduction to valued function fields

Take an extension $(L|K,v)$ such that $vK \prec_\exists vL$ and $Kv \prec_\exists Lv$; we wish to show that $(K,v) \prec_\exists (L,v)$. For this it suffices to show that (K,v) is existentially closed in every finitely generated subextension of $(L|K,v)$ (because every existential sentence only talks about finitely many elements). Recall that finitely generated field extensions are function fields. So we have to prove that for each function field $F|K$ with $F \subseteq L$, (K,v) is existentially closed in (F,v). Since $vK \prec_\exists vL$ and $Kv \prec_\exists Lv$, we also have that $vK \prec_\exists vF$ and $Kv \prec_\exists Fv$. Our task will be achieved if we are able to prove an embedding lemma.

Embedding lemmas for valued fields

Take a highly enough saturated elementary extension (K^*, v^*) of (K, v). Recall that elementary extension means that (K^*, v^*) satisfies the same elementary sentences, with parameters from K, as (K, v). Saturation can be thought of as some sort of "model theoretic richness" or "density" (which has no influence on the elementary properties).

Assume that we are able to show that (F, v) admits a valuation preserving embedding in (K^*, v^*) over K (i.e., leaving the elements of K fixed). Then every existential sentence that holds in (F, v) will also hold in its image in (K^*, v^*), and as existence in a subfield of (K^*, v^*) also means existence in (K^*, v^*), it will also hold there. Finally, as (K^*, v^*) is an elementry extension of (K, v), it will also hold in (K, v), as desired.

Embedding lemmas for valued fields

By general model theory, our assumption that $vK \prec_\exists vF$ and $Kv \prec_\exists Fv$ implies the existence of embeddings of vF in v^*K^* over vK and of Fv in K^*v^* over Kv. So our task is to lift these embeddings to an embedding of (F, v) in (K^*, v^*).

Available tools

Tools we can employ for the construction of such embeddings:

• The embeddings of vF and Fv, via Abhyankar valuations on suitable subfields.

• Under certain restrictive conditions, Ostrowski's and Kaplansky's theory of pseudo Cauchy sequences, as developed in [Ka 1942].

• Hensel's Lemma, which provides criteria for polynomials over henselian fields to admit zeros. Note that the fields (K, v) we are dealing with are henselian (otherwise we have no chance), and as (K^*, v^*) is an elementary extension of (K, v), it is also henselian.

In general, this is all that is presently available to us. When does it suffice?

Structure of the valued function field

In other words, we need that our valued function $(F|K,v)$ has a suitable structure: it (essentially) needs to admit a transcendence basis $\mathcal{T}$ such that we can construct an embedding of $(K(\mathcal{T}),v)$ in (K^*,v^*) by using the first and second tool, and such that the remaining finite extension $(F|K(\mathcal{T}),v)$ is governed by Hensel's Lemma which allows us to extend the embedding of $K(\mathcal{T})$ to an embedding of F. (This means that the finite extension shoud be étale - but we will not use this notion, and instead, after some preparation, introduce a different notion. But before we do this, let us say more about the first and the second tool.

We will now present a result that is very important for us. For the technical, but easy, proof, see Bourbaki, *Commutative Algebra*, Chapter VI, §10.3, Theorem 1.

Let $(F|K,v)$ be an extension of valued fields. Take elements $x_i, y_j \in F$, $i \in I$, $j \in J$, such that the values vx_i, $i \in I$, are rationally independent over vK, and the residues y_jv, $j \in J$, are algebraically independent over Kv. Then the elements x_i, y_j, $i \in I$, $j \in J$, are algebraically independent over K.

Moreover, if we write

$$f = \sum_k c_k \prod_{i\in I} x_i^{\mu_{k,i}} \prod_{j\in J} y_j^{\nu_{k,j}} \in K[x_i, y_j \mid i \in I, j \in J]$$

in such a way that for every $k \neq \ell$ there is some i such that $\mu_{k,i} \neq \mu_{\ell,i}$ or some j such that $\nu_{k,j} \neq \nu_{\ell,j}$, then

$$vf = \min_k v\, c_k \prod_{i\in I} x_i^{\mu_{k,i}} \prod_{j\in J} y_j^{\nu_{k,j}} = \min_k vc_k + \sum_{i\in I} \mu_{k,i} vx_i \,. \qquad (1)$$

That is, the value of the polynomial f is equal to the least of the values of its monomials.

In particular, this implies:

$$\begin{aligned} vK(x_i, y_j \mid i \in I, j \in J) &= vK \oplus \bigoplus_{i \in I} \mathbb{Z} v x_i \\ K(x_i, y_j \mid i \in I, j \in J)v &= Kv\,(y_j v \mid j \in J)\,. \end{aligned}$$

Moreover, the valuation v on $K(x_i, y_j \mid i \in I, j \in J)$ is uniquely determined by its restriction to K, the values vx_i and the residues $y_j v$.

Conversely, if (K, v) is any valued field and we assign to the vx_i any values in an ordered group extension of vK which are rationally independent, then (1) defines a valuation on F, and the residues $y_j v$, $j \in J$, are algebraically independent over Kv.

Important consequences

This lemma has several important consequences:

• It proves the Abhyankar Inequality.

• If $(F|K,v)$ is a valued function field and if equality holds in the Abhyankar Inequality, then the extensions $vF|vK$ and $Fv|Kv$ are finitely generated.

• If $(F|K,v)$ is a valued function field and we choose the x_i and y_j as in the Bourbaki Lemma, then because of the uniqueness assertion of that lemma, the embeddings of vF in v^*K^* and of Fv in K^*v^* give rise to a valuation preserving embedding of $K(x_i, y_j \mid i \in I, j \in J)$ in K^*.

The case of valued function fields with Abhyankar valuations

Assume that $(F|K,v)$ is a valued function field with v an Abhyankar valuation. Then the x_i and y_j form a transcendence basis of $F|K$. Therefore, for $\mathcal{T} = \{x_i, y_j \mid i \in I, j \in J\}$ we obtain the valuation preserving embedding of $K(\mathcal{T})$ in K^*, and $F|K(\mathcal{T})$ is finite. Moreover, $vF|vK$ is finitely generated, and since $vK \prec_\exists vF$, we have that vF/vK is torsion free, hence a finite product of copies of $\mathbb{Z}$. Consequently, the x_i can be chosen such that their values already generate all of vF.

The case of valued function fields with Abhyankar valuations

Likewise, $Fv|Kv$ is finitely generated, and since $Kv \prec_{\exists} Fv$, it is separable, hence separably generated. We can therefore choose the elements y_j such that their residues $y_j v$ form a separating transcendence basis of $Fv|Kv$. Note that in general they will not generate all of Fv, but the remaining finite extension is separable. Together with what we are going to discuss soon, this is the essential part of the proof for our AKE$^{\exists}$ Principle for tame fields.

The general case of valued function fields

If the valuation v on $F|K$ is not Abhyankar, our approach is to first embed a subfunction field with Abhyankar valuation of maximal transcendence degree over K. In order to do so, and then reduce the remaining extension to steps of transcendence degree 1, we use what we call slicing:

Lemma

Take a tame (or separably tame) field (L, v) and a relatively algebraically closed subfield $K \subset L$. If in addition $Lv|Kv$ is an algebraic extension, then K is also a tame (or separably tame) field and moreover, vL/vK is torsion free and $Kv = Lv$.

Note that without the condition on $Lv|Kv$, one does not obtain that vL/vK is torsion free, not even if char $Kv = 0$ (a counterexample is given in [K 2004]).

The case of immediate extensions of transcendence degree 1

Let us give a sketch of the second tool, which uses pseudo Cauchy sequences. In what follows, take $(K(x)|K,v)$ to be a nontrivial immediate extension. Then the set

$$v(x-K) \;=\; \{v(x-c) \mid c\in K\}$$

has no largest element. To show this, take any $c\in K$. Since $vK(x)=vK$, there is $c_1\in K$ such that $vc_1=v(x-c)$, so that $vc_1^{-1}(x-c)=0$. Since $K(x)v=Kv$, there is $c_2\in K$ such that $c_2v=c_1^{-1}(x-c)v$, so that $v(c_1^{-1}(x-c)-c_2)>0$. This implies

$$v(x-c-c_1c_2) \;>\; vc_1 \;=\; v(x-c)\,,$$

and we set $d:=c+c_1c_2$ to obtain $v(x-d)>v(x-c)$.

Pseudo Cauchy sequences

We are going to unravel the information that is contained in $v(x-K)$. A sequence $(a_\nu)_{\nu<\lambda}$ of elements in K (where λ is a limit ordinal) is called a pseudo Cauchy sequence if for all $\rho<\sigma<\tau<\lambda$ we have that

$$v(a_\rho - a_\sigma) \;<\; v(a_\sigma - a_\tau)\,.$$

An element a in some valued field extension (L,v) of (K,v) is called a (pseudo) limit of $(a_\nu)_{\nu<\lambda}$ if for all $\rho<\lambda$, we have that

$$v(a-a_\rho) \;=\; v(a_{\rho+1}-a_\rho)\,.$$

Since the values $v(a_\rho - a_\sigma)$ may be bounded from above in vK, limits of pseudo Cauchy sequences are in general not unique.

Immediate extensions

Using the fact that the set $v(x-K)$ has no largest element, Kaplansky shows:

Theorem (Theorem 1 of [Ka 1942])

There is a pseudo Cauchy sequence in K which does not have a limit in K but admits x as a limit.

Remark: Different pseudo Cauchy sequences can have the same limits. Like for Cauchy sequences, one then calls them equivalent. In [KV 2014] and in [K 2022], approximation types are introduced that correspond to such equivalence classes. They can be thought of as filters consisting of ultrametric balls.

Pseudo Cauchy sequences or approximation types can be used to construct immediate extensions.

Pseudo Cauchy sequences

A pseudo Cauchy sequence $(a_\nu)_{\nu<\lambda}$ is said to be of transcendental type if for every polynomial f over K, the value $vf(a_\nu)$ is fixed for all large enough ν. Otherwise, $(a_\nu)_{\nu<\lambda}$ is said to be of algebraic type. Kaplansky shows:

Theorem (Theorem 2 of [Ka 1942])

If $(a_\nu)_{\nu<\lambda}$ is of transcendental type and x is transcendental over K, then there is an extension of v from K to $K(x)$ such that $(K(x)|K,v)$ is immediate and x is a limit of $(a_\nu)_{\nu<\lambda}$.

If $(K(y)|K,v)$ is another extension such that y is a limit of $(a_\nu)_{\nu<\lambda}$, then $x \mapsto y$ induces a valuation preserving isomorphism from $(K(x),v)$ onto $(K(y),v)$ over K, and consequently, also y is transcendental over K.

This theorem is the tool used in our embedding lemmas. The saturation of (K^*,v^*) is used to find the element $y \in K^*$.

Further, Kaplansky shows:

Theorem (Theorem 3 of [Ka 1942])

If $(a_\nu)_{\nu<\lambda}$ is of algebraic type without a limit in K, f is a polynomial of minimal degree whose value is not ultimately fixed, and a is a root of f, then there is an extension of v from K to $K(a)$ such that $(K(a)|K,v)$ is immediate and a is a limit of $(a_\nu)_{\nu<\lambda}$.

If b is another root of f and b is a limit of $(a_\nu)_{\nu<\lambda}$ under some extension w of v from K to $K(b)$, then $a \mapsto b$ induces a valuation preserving isomorphism from $(K(a),v)$ onto $(K(b),w)$ over K.

Maximality and algebraic maximality

Using these theorems, one can prove:

Theorem (Theorem 4 of [Ka 1942])

A valued field (K, v) is maximal if and only if every pseudo Cauchy sequence in K has a limit in K.

Theorem

A valued field (K, v) is algebraically maximal if and only if every pseudo Cauchy sequence of algebraic type in K has a limit in K.

Note that if (K, v) is not algebraically maximal, then x may be limit of a pseudo Cauchy sequence of algebraic type in K, even if it is transcendental over K.

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More detailed information

A lecture series on valued function fields and the defect can be found on the web page

https://math.usask.ca/fvk/Fvkls.html.

Preprints and further information:

The Valuation Theory Home Page
http://math.usask.ca/fvk/Valth.html.