Tame fields and beyond, I

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Two classes of (quite similar) valuations

Take any prime number p. Every nonzero element

$$a = \frac{r}{s} \in \mathbb{Q}\,,\ \text{where } r,s \in \mathbb{Z},\ s \neq 0\,,$$

can be rewritten as

$$a = p^{\nu}\,\frac{\tilde{r}}{\tilde{s}}\,,\ \text{where }\ \nu,\tilde{r},\tilde{s} \in \mathbb{Z}\,,\ p \nmid \tilde{r},\ p \nmid \tilde{s}\,.$$

Then we set

$$v_p a \ := \ \nu\,.$$

The function v_p is called the p-adic valuation on $\mathbb{Q}$. We set $v_p 0 := \infty$.

Further,

$$|a|_p := p^{-v_p a}$$

is the p-adic absolute value.

Taking the completion of $\mathbb{Q}$ with respect to the topology induced by $|.|_p$, we obtain $\mathbb{Q}_p$, which is called the field of p-adic numbers. The canonical extension of v_p to $\mathbb{Q}_p$ will again be denoted by v_p . Contrary to what holds in the completion $\mathbb{R}$ of $\mathbb{Q}$ with respect to the topology induced by the usual absolute value $|x|$, in $\mathbb{Q}_p$ two integers are close to each other if their difference is divisible by a high power of p.

Two classes of (quite similar) valuations

Take any field K and let $K(t)$ be the rational function field over K. Every element

$$r(t) = \frac{f(t)}{g(t)} \in \mathbb{Q}\,, \text{ where } f,g \in K[t],\, g \neq 0\,,$$

can be rewritten as

$$r(t) = t^{\nu}\,\frac{\tilde{f}(t)}{\tilde{g}(t)}\,, \text{ where } \nu \in \mathbb{Z}\,, \tilde{f},\tilde{g} \in K[t]\,,\, t \nmid \tilde{f},\, t \nmid \tilde{g}\,.$$

Then we set

$$v_t r(t) \;:=\; \nu\,.$$

The function v_t is called the t-adic valuation on $K(t)$. We set $v_t 0 := \infty$.

Two classes of (quite similar) valuations

Taking the completion of $K(t)$ with respect to the topology induced by v_t, we obtain $K((t))$, which is called the field of (formal) Laurent series. The canonical extension of v_t to $K((t))$ will again be denoted by v_t .

This completion can be presented as a power series field:

$$K((t)) = \left\{ \sum_{i=N}^{\infty} c_i t^i \mid N \in \mathbb{Z}\,,\ c_i \in K \right\} .$$

Then

$$v_t \sum_{i=N}^{\infty} c_i t^i = \min\{i \mid c_i \neq 0\}.$$

Valuations

In general, a valuation on a field F is a function from F to an ordered abelian group together with ∞ such that
$v(x) = \infty \Leftrightarrow x = 0$,
$v(xy) = v(x) + v(y)$,
$v(x+y) \geq \min\{v(x), v(y)\}$.

The value group of v is $vF := v(F^\times)$,
and its valuation ring is $\mathcal{O}_v := \{x \in F \mid v(x) \geq 0\}$.
The unique maximal ideal of $\mathcal{O}_v$ is $\mathcal{M}_v := \{x \in F \mid v(x) > 0\}$.
The field $Fv := \mathcal{O}_v / \mathcal{M}_v$ is called the residue field of v.
The residue map of v is the canonical epimorphism $\mathcal{O}_v \to Fv$; extending it to all of F by sending all elements outside of $\mathcal{O}_v$ to ∞ yields the place associated to v, denoted by P_v or simply P.
We set $FP := Fv$.

By (F, v) we denote a field with a valuation v.

The invariants of valued fields

The value group vF and the residue field Fv are called the invariants of the valued field (F,v).

Both $(\mathbb{Q},v_p)$ and $(K(t),v_t)$ have value group $\mathbb{Z}$.

Valuation ring and valuation ideal of v_p on $\mathbb{Q}$ are

$$\mathcal{O}_{v_p} = \left\{\frac{m}{n}\in\mathbb{Q} \mid p\nmid n\right\} \quad \text{and} \quad \mathcal{M}_{v_p} = \left\{\frac{m}{n}\in\mathbb{Q} \mid p\nmid n\,,\, p|m\right\}.$$

Consequently, the residue field of $(\mathbb{Q},v_p)$ is $\mathbb{Z}/p\mathbb{Z}=\mathbb{F}_p$.
Valuation ring and valuation ideal of v_t on $K(t)$ are

$$\mathcal{O}_{v_t} = \left\{\frac{f}{g}\in\mathbb{Q} \mid t\nmid g\right\} \quad \text{and} \quad \mathcal{M}_{v_t} = \left\{\frac{f}{g}\in\mathbb{Q} \mid t\nmid g\,,\, t|f\right\}.$$

Since for every $f\in K[t]$ there is some $c\in K$ such that $t|(f-c)$, the residue field of $(K(t),v_t)$ is K.

The invariants of $\mathbb{Q}_p$ and $K((t))$

If (K, v) is a valued field and $L|K$ is a field extension, then there is always an extension of v to L. By $(L|K, v)$ we will denote a valued field extension, where L is endowed with the valuation v, and K with its restriction. The extension $(L|K, v)$ is called immediate if the canonical embeddings

$$vK \hookrightarrow vL \quad \text{and} \quad Kv \hookrightarrow Lv$$

are onto (or in short: $vK = vL$ and $Kv = Lv$).

Completions are always immediate extensions. Hence, both $(\mathbb{Q}_p, v_p)$ and $(K((t)), v_t)$ have value group $\mathbb{Z}$, $(\mathbb{Q}_p, v_p)$ has residue field $\mathbb{F}_p$, and $(K((t)), v_t)$ has residue field K.

Comparison of $\mathbb{Q}_p$ with $\mathbb{F}_p((t))$

For now, we are particularly interested in the case of $K = \mathbb{F}_p$ (as a representative of any finite field).

We have:

1) $v_p\mathbb{Q}_p = \mathbb{Z} = v_t\mathbb{F}_p((t))$,

2) $\mathbb{Q}_p v_p = \mathbb{F}_p = \mathbb{F}_p((t))v_t$,

3) both valued fields are complete.

However, $\operatorname{char}\mathbb{Q}_p = 0$, while $\operatorname{char}\mathbb{F}_p((t)) = p$.

Comparison of $\mathbb{Q}_p$ with $\mathbb{F}_p((t))$: decidability

In 1965, J, Ax and S. Kochen [AxKo 1965], and independently Yu. Ershov [E 1965/66/67], showed that the elementary theory $\mathrm{Th}(\mathbb{Q}_p, v_p)$ of $(\mathbb{Q}_p, v_p)$ is decidable. The question whether the same holds for the elementary theory of $(\mathbb{F}_p((t)), v_t)$ has remained open. It is worthwhile mentioning that this problem has connections with the equally open problem of local uniformization (a local form of resolution of singularities) in positive characteristic.

How can we prove decidability of a theory? One way is to present a complete recursive axiomatization.

As the property of of a valued field to be complete is not elementary, we have to work with a property that represents its elementary content.

Henselian fields

A valued field (F, v) is called henselian if v admits a unique extension to every algebraic extension field, or equivalently, if it satisfies Hensel's Lemma:

For every polynomial $f \in \mathcal{O}_K[X]$ the following holds: if $b \in \mathcal{O}_v$ satisfies

$$vf(b) > 0 \quad \text{and} \quad vf'(b) = 0\,, \tag{1}$$

then f admits a root $a \in \mathcal{O}_K$ such that $v(a-b) > 0$.

This was in fact a lemma proven by Hensel for $\mathbb{Q}_p$, but now it is also used as a property of valued fields, which is elementary.

Every complete valued field of rank 1, i.e., whose value group is archimedean, or in other words, embeddable in $\mathbb{R}$, is henselian. (However, this does not hold in higher rank!)

Axiomatization for $\mathbb{Q}_p$

Here is a complete recursive axiomatization for $(\mathbb{Q}_p, v_p)$:

(K, v) is a henselian valued field with $v_p p$ the smallest positive element of its value group, which is a $\mathbb{Z}$-group, and its residue field is $\mathbb{F}_p$.

An ordered abelian group Γ is a $\mathbb{Z}$-group if it is elementarily equivalent to $\mathbb{Z}$, or equivalently, it has $\mathbb{Z}$ as a convex subgroup such that $\Gamma/\mathbb{Z}$ is divisible.

It was already known early on that a direct adaptation of this axiom system to $(\mathbb{F}_p((t)), v_p)$, replacing the condition on $v_p p$ by "char $K = p$", is not complete. There is an elementary property that is automatically satisfied by p-adic valuations, but not by t-adic valuations.

The Lemma of Ostrowski and the defect

An extension $(L|K,v)$ is called unibranched if the extension of v from K to L is unique. For a finite unibranched extension $(L|K,v)$, the Lemma of Ostrowski says:

$$[L:K] \;=\; \tilde{p}^{\nu}\cdot(vL:vK)[Lv:Kv]\,, \tag{2}$$

where ν is a non-negative integer and $\tilde{p}$ is the characteristic exponent of Kv, that is, $\tilde{p} = \mathrm{char}\, Kv$ if it is positive and $\tilde{p} = 1$ otherwise.

The factor $d(L|K,v) := \tilde{p}^{\nu}$ is the defect of the extension $(L|K,v)$. We call $(L|K,v)$ a defect extension if $d(L|K,v) > 1$, and a defectless extension if $d(L|K,v) = 1$. Nontrivial defect only appears when $\mathrm{char}\, Kv = p > 0$, in which case $\tilde{p} = p$.

The defect is a main obstruction in the model theory of valued fields and for local uniformization in positive characteristic.

Defectless fields

A henselian field (F, v) is defectless if no finite extension of (K, v) has nontrivial defect. Every henselian field (K, v) with char $Kv = 0$ is defectless. The field $\mathbb{Q}_p$ of p-adic numbers is defectless.

Examples for valued fields that are not defectless:

1) certain infinite algebraic extensions of $\mathbb{Q}_p$, such as $\mathbb{Q}_p^{ab}$,

2) if (K, v) is a nontrivially valued field that is not perfect, then its separable-algebraic closure is henselian, but not defectless,

3) the perfect hull $\mathbb{F}_p((t))^{1/p^\infty}$ of $\mathbb{F}_p((t))$ is not defectless (as we will see now).

Abhyankar's Example

The extension of v_t from $\mathbb{F}_p((t))$ to the purely inseparable extension $K := \mathbb{F}_p((t))^{1/p^\infty}$ is unique and will again be denoted by v_t . Take a root ϑ of the Artin-Schreier polynomial

$$X^p - X - \frac{1}{t}\,.$$

As an algebraic extension of the henselian field $(\mathbb{F}_p((t)), v_t)$, also (K, v_t) is henselian. Therefore, the extension $(K(\vartheta)|K, v_t)$ is unibranched. It is also immediate, so

$$(v_t K(\vartheta) : v_t K)[K(\vartheta)v_t : Kv_t] = 1 \cdot 1 = 1\,.$$

Hence by the Lemma of Ostrowski,

$$d(K(\vartheta)|K, v_t) \;=\; [K(\vartheta) : K] \;=\; p\,.$$

The first examples for defect extensions were inseparable, but Abhyankar's Example shows that nontrivial defect can also appear over perfect fields. For a large collection of other examples of defect extensions, see [K 2011].

It is worthwhile to note that Abhyankar's intention was not to give an example of a defect extension. Instead, he showed by his example that Puiseux exponents in algebraic extensions in positive characteristic need not have a common denominator. In fact, while working implicitly with defects in his attempts to prove resolution of singularities in positive characteristic, he never addressed the defect directly.

Axioms for $\mathbb{F}_p((t))$

It can be shown that all power series fields with their canonical valuations, hence also $(\mathbb{F}_p((t)), v_t)$, are defectless fields. This suggests the following axiom system for $(\mathbb{F}_p((t)), v_t)$:

(K, v) is a henselian defectless field of characteristic p whose value group is a $\mathbb{Z}$-group, and whose residue field is $\mathbb{F}_p$.

I will call this the naive axiom system, because unfortunately it turned out that it is not complete (see [K 2004]). More about this later. The search for a complete recursive axiom system is still on.

On the way to $\mathbb{F}_p((t))$

Let us list the classes of valued fields of positive characteristic for which some good model theoretic results were known by the 1980's.

1) Algebraically closed valued fields (Abraham Robinson);

2) Algebraically maximal Kaplansky fields.

A valued field is called algebraically maximal if it does not admit proper immediate algebraic extensions. Note that for Kaplansky fields, "algebraically maximal" is equivalent to "henselian and defectless".

Kaplansky fields

Kaplansky fields were defined by Irving Kaplansky in [Ka 1942] by means of his famous (but slightly mysterious) "hypothesis A". He proved that the maximal immediate extensions of such a valued field are unique up to isomorphism over the field. The existence of maximal immediate extensions of arbitrary valued fields had been proven by W. Krull in [Kr 1932]; the proof was later improved by K. A. H. Gravett in [G 1956].

Several authors worked on unravelling and understanding hypothesis A. The ultimate analysis was developed by M. Pank in his PhD thesis supervised by P. Roquette, see [KPR 1986]. It was shown that Kaplansky fields (K, v) can be axiomatized by:

(KAP1) if char $Kv = p > 0$, then vK is p-divisible,

(KAP2) if char $Kv = p > 0$, then Kv does not admit finite extensions whose degrees are divisible by p.

Kaplansky fields

In particular, the residue field Kv of a Kaplansky field (K, v) is perfect, and if in addition (K, v) is algebraically maximal, then also K is perfect. However, axiom (KAP2) asks for more, namely, that Kv also does not have separable extensions whose degree is divisible by p. This is a condition that significantly limits the number of possible applications. On the other hand, when we drop this part of axiom (KAP2), we lose the uniqueness.

Some model theoretic results about algebraically maximal Kaplansky fields have been proved; we will come back to this later.

The next step: tame fields

A more natural and applicable class of valued fields is that of tame fields. A valued field (K, v) is a tame field if it is henselian and satisfies:

(T1) if char $Kv = p > 0$, then vK is p-divisible,

(T2) Kv is perfect,

(T3) (K, v) is a defectless field.

Note that compared to algebraically maximal Kaplansky fields, the condition on the residue field has been relaxed. As now we are missing the uniqueness of maximal immediate extensions, proving model theoretic results is considerably more difficult. Let us give a survey on what is known about the model theory of tame fields.

Ax-Kochen-Ershov Principles

J. Ax and S. Kochen [AxKo 1965] proved (a corrected version of) Artin's Conjecture. To this end, they showed that if (K,v) and (L,v) are henselian fields with residue fields of characteristic 0, then they satisfy the following Ax–Kochen–Ershov Principle:

$$vK \equiv vL \;\wedge\; Kv \equiv Lv \;\Longrightarrow\; (K,v) \equiv (L,v)\,. \tag{3}$$

We call this the AKE$^{\equiv}$ Principle. The following analogue for elementary extensions will be called the AKE$^{\prec}$ Principle:

$$(K,v) \subseteq (L,v) \;\wedge\; vK \prec vL \;\wedge\; Kv \prec Lv \;\Longrightarrow\; (K,v) \prec (L,v)\,. \tag{4}$$

Also this principle has been proven in the case where (K,v) and (L,v) are henselian fields with residue fields of characteristic 0. By the 1980's, these principles were also known to hold for other classes of valued fields (algebraically closed, p-adic, finitely ramified, algebraically maximal Kaplansky).

A third AKE Principle

If $\mathcal{M}$ is an $\mathcal{L}$-structure and $\mathcal{M}'$ a substructure of $\mathcal{M}$, then we will say that $\mathcal{M}'$ is existentially closed in $\mathcal{M}$ and write $\mathcal{M}' \prec_\exists \mathcal{M}$ if every existential $\mathcal{L}$-sentence with parameters from $\mathcal{M}'$ which holds in $\mathcal{M}$ also holds in $\mathcal{M}'$. For the meaning of "existentially closed in" in the setting of fields and of ordered abelian groups, see [KPr 1984]. This notion is important for the proof of theorems like Nullstellensätze.

The following will be called the AKE$^\exists$ Principle:

$$(K,v) \subseteq (L,v) \;\wedge\; vK \prec_\exists vL \;\wedge\; Kv \prec_\exists Lv \;\Rightarrow\; (K,v) \prec_\exists (L,v)\,. \tag{5}$$

While it was usually not considered explicitly by other authors, it was commonly known that this principle holds whenever (K,v) belongs to one of the above mentioned classes.

AKE Principles for tame fields

We say that (K, v) is an equicharacteristic valued field if $\mathrm{char}\, K = \mathrm{char}\, Kv$, and that it is a mixed characteristic valued field if $\mathrm{char}\, K = 0$ and $\mathrm{char}\, Kv > 0$. The following theorem was proven in [K 2016]:

Theorem

a) Every extension (L, v) of a tame field (K, v) satisfies the $\mathrm{AKE}^{\exists}$ Principle.

b) Every extension $(L|K, v)$ of tame fields (K, v) and (L, v) satisfies the $\mathrm{AKE}^{\prec}$ Principle.

c) Every two equicharacteristic tame fields (K, v) and (L, v) satisfy the $\mathrm{AKE}^{\equiv}$ Principle.

Negative or unknown results

It has been shown in [AnK 2016] that the $AKE^{\equiv}$ Principle does not hold for mixed characteristic tame fields.

It is not known whether tame fields admit quantifier elimination in a suitable language. It seems highly unlikely that the various languages that so far have worked for other classes of valued fields (RV or AMC structures, Pas language) would work for tame fields. The reason for this is in fact connected with the non-uniqueness of maximal immediate extensions.

Decidability for tame fields

As an immediate consequence of the AKE$^{\prec}$ Principle, we obtain the following criterion for decidability:

Theorem

Let (K, v) be a tame field of equal characteristic. Assume that the theories $\mathrm{Th}(vK)$ *of its value group (as an ordered group) and* $\mathrm{Th}(Kv)$ *of its residue field (as a field) both admit recursive elementary axiomatizations. Then also the theory of (K, v) as a valued field admits a recursive elementary axiomatization and is decidable.*

In order to apply this theorem, we need some preparations.

Henselizations

A henselization of the valued field (K, v) is an algebraic extension of (K, v) which admits a valuation preserving embedding in every other henselian extension of (K, v). Henselizations always exist and are unique up to valuation preserving isomorphism over K; therefore we will talk of *the* henselization of (K, v) and denote it by $(K, v)^h = (K^h, v^h)$.

The henselization of (K, v) is an immediate separable-algebraic extension. Therefore, all algebraically maximal and all maximal valued fields are henselian. A valued field is called maximal if it does not admit any proper immediate extensions.

General power series fields

Take any field K and any ordered abelian group Γ. The power series field with coefficients in K and exponents in Γ, denoted by $K((\Gamma))$, is defined as follows. As a set, it is the collection of all power series

$$a := \sum_{\gamma\in\Gamma} c_\gamma t^\gamma$$

with $c_\gamma \in K$ for which the support

$$\mathrm{supp}(a) := \{\gamma \in \Gamma \mid c_\gamma \neq 0\}$$

is well-ordered. This makes it possible to define multiplication (while addition is componentwise) and to define the t-adic valuation (also called minimum support valuation):

$$v_t\, a := \min \mathrm{supp}(a)\,.$$

Decidability for tame fields

As an application of the decidability result for tame fields, we obtain the following:

Theorem

Take $q = p^n$ for some prime p and some $n \in \mathbb{N}$, and an ordered abelian group Γ. Assume that Γ is divisible or elementarily equivalent to the p-divisible hull of $\mathbb{Z}$. Then $(\mathbb{F}_q((t^\Gamma)), v_t)$ is a tame field, and its elementary theory is decidable.

Applications of the theory of tame fields

The results on tame fields have been applied in [K 2004] to the structure theory of spaces of places of algebraic function fields and to model theoretic questions related to rational places, large fields and local uniformization. They were also applied by S. Anscombe and A. Fehm to the problem of the decidability of the existential theory of $\mathbb{F}_p((t))$. We will come back to some of these topics later.

We will now discuss the two main theorems used for the proofs of the results on tame fields. They were also applied to prove results on local uniformization in positive caharacteristic.

The Abhyankar Inequality

If Γ is any abelian group, then the rational rank of Γ is $\mathrm{rr}\,\Gamma := \dim_{\mathbb{Q}} \mathbb{Q} \otimes \Gamma$. This is the maximal number of rationally independent elements in Γ.
If $(F|K,v)$ is an arbitrary valued field extension of finite transcendence degree, then we have the Abhyankar inequality:

$$\mathrm{trdeg}\, F|K \;\geq\; \mathrm{rr}\; vF/vK \;+\; \mathrm{trdeg}\, Fv|Kv\,. \tag{6}$$

We call v an Abhyankar valuation and its associated place P an Abhyankar place if equality holds in (6). (In this case we also say that $(F|K,v)$ is an extension without transcendence defect.)

Note: if v is trivial on K, then $vK = 0$ and $Kv \simeq K$.

The Generalized Stability Theorem

The first main theorem is:

Theorem (K, thesis 1989; K 2010)

(Generalized Stability Theorem)
Assume that v is an Abhyankar valuation on the function field $F|K$, not necessarily trivial on K. If (K,v) is a defectless field, then (F,v) is a defectless field.

Here is an application, which shows how useful it can be to work with "existentially closed in" instead of "elementary extension".

Theorem

Take an extension $(L|K,v)$ without transcendence defect of a henselian defectless field (K,v). Then the extension satisfies the $AKE^{\exists}$ Principle.

The Henselian Rationality Theorem

By a function field we mean an algebraic function field, i.e., a finitely generated field extension (usually transcendental).

The second main theorem is:

Theorem ([K 2019])

(Henselian Rationality Theorem)
Let (K, v) be a tame field and (F, v) an immediate function field over (K, v). Assume that $F|K$ is an extension of transcendence degree 1. Then there is $x \in F$ such that $F \subset K(x)^h$.

Note that if the latter holds, then $F^h = K(x)^h$; this explains the name of the theorem.

The theorem is trivial if char $Kv = 0$, but hard to prove otherwise.

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More detailed information

A lecture series on valued function fields and the defect can be found on the web page

https://math.usask.ca/fvk/Fvkls.html.

Preprints and further information:

The Valuation Theory Home Page
http://math.usask.ca/fvk/Valth.html.