Martin's Maximum, Woodin's $\mathbb{P}_{\max}$ axiom (*), and Cantor's Continuum Problem

Hilbert's First Problem revisited

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Cantor's Continuum Problem

- Georg Cantor (1873): While there are as many algebraic real numbers as there are natural numbers, there are in total more real numbers than natural numbers.

Definition

Cantor's Continuum Hypothesis, CH: If $A \subset \mathbb{R}$ is uncountable, then A has as many elements as there are real numbers, or: $2^{\aleph_0} = \aleph_1$.

Cantor (1878): [...] so fragt es sich, in *wie viel* [...] Klassen die linearen Mannigfaltigkeiten zerfallen, wenn Mannigfaltigkeiten von gleicher Mächtigkeit in eine und dieselbe Klasse, Mannigfaltigkeiten von verschiedener Mächtigkeit in verschiedene Klassen gebracht werden. Durch ein Induktionsverfahren, auf dessen Darstellung wir hier nicht näher eingehen, wird der Satz nahe gebracht, daß die Anzahl der [...] sich ergebenden Klassen [...] eine endliche und zwar, daß sie gleich *Zwei* ist.

- David Hilbert's First Problem (1900): Show that CH is true!

In his talk at the IMC in Paris, Hilbert says: Die Untersuchungen von Cantor über solche Punktmengen machen einen Satz sehr wahrscheinlich, dessen Beweis jedoch trotz eifrigster Bemühungen bisher noch Niemanden gelungen ist; dieser Satz lautet: Jedes System von unendlich vielen reellen Zahlen d. h. jede unendliche Zahlen- (oder Punkt)menge ist entweder der Menge der ganzen natürlichen Zahlen 1, 2, 3, ... oder der Menge sämmtlicher reellen Zahlen und mithin dem Continuum, d. h. etwa den Punkten einer Strecke aequivalent [...]

New Axioms

- Kurt Gödel (1938): CH cannot be refuted in ZFC.

In 1947, Gödel wrote: [...] one may on good reason suspect that the role of the continuum problem in set theory will be this, that it will finally lead to the discovery of new axioms which will make it possible to disprove Cantor's conjecture.

- Paul Cohen (1963): CH cannot be proven in ZFC.

In 1966, Cohen wrote: A point of view which the author feels may eventually come to be accepted is that CH is obviously false. [...] $\aleph_1$ is the set of countable ordinals and this is merely a special and the simplest way of generating a higher cardinal. The set C [the continuum] is, in contrast, generated by a totally new and more powerful principle, namely the Power Set Axiom. It is unreasonable to expect that any description of a larger cardinal which attempts to build up that cardinal from ideas deriving from the Replacement Axiom can ever reach C.

New axioms, cont'd

Over the years, various sets of natural axioms emerged which decide questions which were left open by ZFC.

- Large cardinal axioms
- Determinacy hypotheses
- Constructibility (from Gödel's L via core models to Woodin's "ultimate-L")

M. Magidor: If a mathematical object can be imagined in a reasonable way, then it exists!

Forcing axioms, or more generally: Maximality principles, try to formalize this approach.

A delicate point: do you want to maximize the $\Pi_2^{H_{\omega_2}}$ or the $\Sigma_2^{H_{\omega_2}}$ theory? Can't have both. CH is a $\Sigma_2^{H_{\omega_2}}$ statement. The situation with respect to $\Pi_2^{H_{\omega_2}}$ maximality is better understood than the one with respect to $\Sigma_2^{H_{\omega_2}}$ maximality.

Forcing Axioms, cont'd

Definition

Foreman-Magidor-Shelah (1988): Formulated Martin's Maximum, MM, a strengthening of Martin's Axiom, MA: If the forcing $\mathbb{P}$ preserves stationary subsets of ω_1 and if $\mathcal{D}$ is a family of $\aleph_1$ many sets which are all dense in $\mathbb{P}$, then there is a $\mathcal{D}$-generic filter.

MM gives many natural answers to questions which are undecidable on the basis of ZFC, e.g.:

- There is a non-free Whitehead group (Shelah 1974).
- Kaplansky's Conjecture holds true (Solovay-Woodin 1976).
- Every automorphism of the Calkin algebra of a separable Hilbert space is inner (Farah 2011).
- Friedman's Problem (Foreman-Magidor-Shelah 1988).

The classical way to force MM is start with a model of ZFC plus a supercompact cardinal, δ, and perform a semi-proper iteration of length δ. A reflection principle will hold in the generic extension which verifies full MM. In fact, a strengthening of MM may be arranged to hold in the extension:

Definition

Foreman-Magidor-Shelah (1988): Martin's Maximum^{++}, MM^{++}: If the forcing $\mathbb{P}$ preserves stationary subsets of ω_1, if $\mathcal{D}$ is a family of $\aleph_1$ many sets which are all dense in $\mathbb{P}$, and if $\{\tau_i : i < \omega_1\}$ is a collection of names for stationary subsets of ω_1, then there is a $\mathcal{D}$-generic filter such that $\tau_i^g = \{\xi : \exists p \in g\, p \Vdash \xi \in \tau_i\}$ is stationary for each $i < \omega_1$.

The $\mathbb{P}_{\mathrm{max}}$-axiom $(*)$

Definition

Woodin (1990's): Formulates $(*)$, a maximality principle for sets of size $\aleph_1$: The Axiom of Determinacy holds in $L(\mathbb{R})$, and there is a filter $g \subset \mathbb{P}_{\mathrm{max}}$ which is generic over $L(\mathbb{R})$ such that $H_{\omega_2} \subset L(\mathbb{R})[g]$.

$(*)$ is complete (in Ω-logic) with respect to questions about H_{ω_2}:

- $\boldsymbol{\delta}^1_2 = \omega_2$.
- ψ_{AC} and ϕ_{AC}, variants of Friedman's Problem.
- Admissible club guessing and the club bounding principle.

The $\mathbb{P}_{\max}$-axiom $(*)$, cont'd

- W.H. Woodin showed that $(*)$ is $\Pi_2^{H_{\omega_2}}$ maximal: in the presence of large cardinals, if a given $\Pi_2^{H_{\omega_2}}$ statement is Ω consistent, then that statement is Ω provable from ZFC plus $(*)$.

The classical way to force $(*)$ is start with a model of ZF plus $V = L(\mathbb{R})$ plus AD (the Axiom of Determinacy) and force with $\mathbb{P}_{\max}$.

Competitors or twins?

What about the size of the continuum?

Theorem

Foreman-Magidor-Shelah (1988) and Woodin (1990's): Both MM *and* $(*)$ *imply that* $2^{\aleph_0} = \aleph_2$.

MM and $(*)$ were known to have many consequences in common, but they were also known to not follow from each other.

Open questions since the mid 1990's: What is the relation between MM and $(*)$? Does Martin's Maximum^{++}, a further strengthening of MM, imply $(*)$? Can $(*)$ be forced over a model of ZFC with a large cardinal? Is (*), like MM, consistent with all large cardinals?

A theorem joint with David Asperó

Theorem

D. Asperó and R. Schindler (2019): Martin's Maximum^{++} implies the $\mathbb{P}_{\text{max}}$ axiom $(*)$.

This result appeared in the May 2021 issue (Volume 193, no. 3, pp. 793-835) of the Annals of Mathematics.

Proof of the theorem with Asperó

Proof.

Fix $A \subset \omega_1$. Let $D \subset \mathbb{P}_{\max}$ be dense, $D \in L(\mathbb{R})$. By MM^{++}, it suffices to show that there is a forcing $\mathbb{P}$ such that

- $\mathbb{P}$ preserves stationary subsets of ω_1, and
- $\mathbb{P}$ forces that there is a $\mathbb{P}_{\max}$ condition $p \in D^{V^{\mathbb{P}}}$ together with a generic iterate p^* of p such that $a^{p^*} = A$ and $I^{p^*} = \mathrm{NS}_{\omega_1}^{V^{\mathbb{P}}} \cap p^*$.

Such a $\mathbb{P}$ may be constructed as an "$\mathcal{L}$-forcing" (partially building upon methods developed by Ronald Jensen). The conditions in $\mathbb{P}$ give finitely much information about the objects to be added plus information about "virtual side conditions." □

The entirely new feature is that the side conditions of $\mathbb{P}$ aren't in the ground model.

$(*)$ is *equivalent* to a schema of Ω consistent $\Pi_2^{H_{\omega_2}}$ statements in the language of set theory augmented by predicates for NS_{ω_1} as well as sets of reals in $L(\mathbb{R})$.

Moreover, by our proof, $(*)$ is in fact - in the presence of large cardinals - also *equivalent* to a bounded form of MM^{++}.

Our proof shows that every *honestly consistent* statement which is Σ_1 in predicates from $H_{\omega_2} \cup \{\mathrm{NS}_{\omega_1}\} \cup (\mathcal{P}(\mathbb{R}) \cap L(\mathbb{R}))$ may be forced by a stationary set preserving forcing.

This might open up a scenario for actually proving Woodin's Ω conjecture.

Extensions of $(*)$

Definition

Woodin: $(*)^{++}$: There is a model $L(\mathbb{R},\Gamma)$ of AD and a filter $g \subset \mathbb{P}_{\max}$ which is generic over $L(\mathbb{R},\Gamma)$ such that $\mathcal{P}(\mathbb{R}) \subset L(\mathbb{R},\Gamma)[g]$.

Woodin: All the known models of MM violate $(*)^{++}$.

Open question: Is $(*)^{++}$ compatible with MM?

Duality

Our result connects the two approaches:

- produce an interesting ZFC model by forcing over a ZFC model with large cardinals, and
- produce an interesting ZFC model by forcing over a ZF model of determinacy

with one another.

By work of Larson, Sargsyan, Woodin, and others, bounded fragments/implications of MM^{++} may be forced over determinacy models.

By work of A. Lietz and myself, $(*)$ may be forced over a model of ZFC plus an inaccessible limit of κ^{++} supercompacts.

Open questions: Can MM^{++} be forced over a determinacy model? Can $(*)$ be forced over a ZFC model with an inaccessible limit of Woodin cardinals? Can "NS_{ω_1} is ω_1 dense" be forced over a ZFC model?

The Continuum Problem

It is an empirical fact that the most sophisticated extensions of ZFC which decide the value of the continuum prove $2^{\aleph_0} \in \{\aleph_1, \aleph_2\}$.

Challenges for future research:

- Explore "V = ultimate-L."
- Embed MM^{++} into a "complete" theory of V.
- In the light of the results of Goldstern-Kellner-Shelah, develop well-justifiable theories which prove that $2^{\aleph_0} > \aleph_2$ and which possibly even produce effective counterexamples to $2^{\aleph_0} \leq \aleph_2$ or which give pairwise different values to as many entries in Cichoń's diagram as possible.

144 years after Cantor formulated it, the Continuum Problem remains one of *the* driving forces of set theoretical research.