Nonstandard models of the reals and symmetrical completeness

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Real numbers and infinitesimals

(Probably) all of us feel comfortable with the ordered field of real numbers; it (more or less) describes the space in which we are living. On the other hand, in analysis we work with the abstract notion of infinitesimals. Nonstandard analysis shows that infinitesimals can actually be seen and handled as concrete elements in larger ordered fields. In fact, they already appear in simple constructions in real algebra: take the rational function field $\mathbb{R}(x)$ in one variable over the reals. The canonical ordering of $\mathbb{R}$ can be extended in many ways to $\mathbb{R}(x)$. These extensions are uniquely determined by the cut that x induces in $\mathbb{R}$ under the given ordering:

$$(\{a \in \mathbb{R} \mid a < x\}, \{b \in \mathbb{R} \mid b > x\}) .$$

If we take the cut $(\mathbb{R}^{\leq 0}, \mathbb{R}^{>0})$, then x will be a positive infinitesimal, i.e., it is smaller than all positive reals.

Cuts

Take a totally ordered set $(T, <)$. A set $S \subseteq T$ is called an initial segment of T if for each $s \in S$ every $t < s$ also lies in S.
Similarly, $S \subseteq T$ is called a final segment of T if for each $s \in S$ every $t > s$ also lies in S.
A pair

$$(\Lambda^L, \Lambda^R)$$

of subsets of T is called a cut in T if Λ^L is an initial segment of T and

$$\Lambda^R = T \setminus \Lambda^L .$$

It then follows that Λ^R is a final segment of T,

$$T = \Lambda^L \cup \Lambda^R ,$$

and

$$\Lambda^L < \Lambda^R ,$$

i.e., $a < b$ for all $a \in \Lambda^L, b \in \Lambda^R$.

Convention

From now on, when we talk of ordered sets, abelian groups and fields, we tacitly assume the orderings to be total. When we talk of cuts, we will mean Dedekind cuts, i.e., upper and lower cut sets are nonempty.

$\mathbb{R}$ is cut complete and real closed

The ordered field $(\mathbb{R}, <)$ is cut complete, i.e., every cut (Λ^L, Λ^R) is principal, which means that Λ^L has a largest or Λ^R has a smallest element. Also $(\mathbb{Z}, <)$ is cut complete.

The only cut complete ordered field is $(\mathbb{R}, <)$.

$(\mathbb{R}, <)$ is also real closed, i.e., no proper algebraic extension admits an ordering.

Power series fields

Take any field K and any ordered abelian group Γ. The power series field with coefficients in K and exponents in Γ, denoted by $K((\Gamma))$, is defined as follows. As a set, it is the collection of all power series

$$a := \sum_{\gamma\in\Gamma} a_\gamma x^\gamma$$

with $a_\gamma \in K$ for which the support

$$\mathrm{supp}(a) := \{\gamma\in\Gamma \mid a_\gamma \neq 0\}$$

is well-ordered. This makes it possible to define multiplication (while addition is componentwise) and to define the x-adic valuation

$$v_x a := \min \mathrm{supp}(a)\,.$$

Ordered power series fields

If $(K, <)$ is an ordered field, then its ordering can be extended to an ordering of $K((\Gamma))$: take $a \in K((\Gamma))$ and set $\gamma_0 := \min \operatorname{supp}(a)$; then define

$$a > 0 \;:\Leftrightarrow\; a_{\gamma_0} > 0\,.$$

Theorem

$(K((\Gamma)), <)$ is real closed if and only if $(K, <)$ is real closed and Γ is divisible (e.g. $\Gamma = \mathbb{Q}$ or $\Gamma = \mathbb{R}$).

Hence if Γ is divisible, then $(\mathbb{R}((\Gamma)), <)$ is an elementary extension of $(\mathbb{R}, <)$.

What about an exponential function?

Can we extend the exponential function from $\mathbb{R}$ to $\mathbb{R}((\Gamma))$? In [KKsS 1997] it was shown:

Theorem

If $\Gamma \neq \{0\}$, then $\mathbb{R}((\Gamma))$ does not admit exponential functions which have the same elementary properties as the exponential function on the reals.

What about the Banach Fixed Point Theorem?

Can we realize certain properties of the reals that are not elementary?

A fixed point of a function $T : X \to X$ is an element $x \in X$ such that $T(x) = x$. The following theorem holds for $X = \mathbb{R}$ with the usual metric $d(x, y) = |x - y|$:

Theorem (Banach Fixed Point Theorem)

Let (X, d) be a nonempty complete metric space with a function $T : X \to X$. Assume that there exists $q \in [0, 1)$ such that

$$d(T(x), T(y)) \leq qd(x, y)$$

for all $x, y \in X$. Then T admits a unique fixed point.

However, completeness is not an elementary property and does not necessarily hold in elementary extensions of the reals.

Ultrametric balls

The power series field $K((\Gamma))$ has the following distinguished subsets: for each $\gamma \in \Gamma$,

$$B_\gamma(0) \ := \ \{a \in K((\Gamma)) \mid \min \operatorname{supp}(a) \geq \gamma\} \, .$$

These sets all include 0, and they form a totally ordered set (a chain) under inclusion. Further, we set

$$B_\gamma(a) \ := \ a + B_\gamma(0) \, .$$

The collection

$$\{B_\gamma(a) \mid a \in K((\Gamma)) \, , \, \gamma \in \Gamma\}$$

is the ultrametric ball space underlying $K((\Gamma))$.

Ultrametric distance

We can define an ultrametric distance by setting

$$u(a,b) \;:=\; B_\gamma(a-b) \quad \text{where} \quad \gamma = \min \operatorname{supp}(a-b)$$

for $a \neq b$, and $u(a,a) := \emptyset$.

Spherical completeness

Also an abstract notion of ultrametrics $u(.,.)$, not referring to $K((\gamma))$, can be defined, giving rise to an associated ultrametric ball space.

An ultrametric ball space is called spherically complete if every chain of nonempty balls has a nonempty intersection.

Theorem

For every power series field $K((\Gamma))$, the underlying ultrametric ball space is spherically complete.

Note: every valuation gives rise to an ultrametric distance. The p-adic distance is an ultrametric, and the underlying ultrametric ball space is spherically complete.

The ultrametric Banach Fixed Point Theorem

In 1990 Sibylla Prieß-Crampe proved an ultrametric version of the Banach Fixed Point Theorem:

Theorem

Take an ultrametric u on a set X whose underlying ball space is spherically complete. If $T : X \to X$ satisfies

$$u(T(x), T(y)) < u(x, y)$$

for all $x, y \in X$, then T admits a unique fixed point.

However...

... this is not what we are after. Already the fact that in an ultrametric at least two sides of a triangle are always equal and every element of an ultrametric ball is its center, shows that what we just presented is not a *real* Banach Fixed Point Theorem.

Ball spaces

The notion of spherical completeness is too powerful to be restricted to ultrametric ball spaces. Generalizing, we call any nonempty set S together with a collection of nonempty subsets a ball space, and we call it spherically complete if every chain of balls has a nonempty intersection.

This concept has been exploited to prove fixed point theorems in various settings, see [KK 2015], [KKP 2015], [KK 2017], [KKsSo 2017], [BlCLSz 2019], [CKK 2021]. For the further study of ball spaces, see [KuK 2019] and [BaKK 2021].

The order ball space

Now we can consider the order ball space of any ordered field, ordered abelian group or ordered set by taking the balls to be the intervals $[a, b]$ with $a \leq b$. Is this ball space spherically complete for all $K((\Gamma))$?

Given a cut (Λ^L, Λ^R) in an ordered set, we can consider intervals $[a, b]$ with $a \in \Lambda^L$ and $b \in \Lambda^R$. Can we not "zoom in" on cuts by chains of intervals and deduce that spherical completeness of the order ball space implies that the ordered set is cut complete, showing that the only ordered field with spherically complete order ball space is $\mathbb{R}$?

Cofinality of a cut and symmetrical completeness

The cofinality of a cut (Λ^L, Λ^R) is the pair (κ, λ) where κ is the cofinality of Λ^L and λ is the coinitiality of Λ^R. Recall that the coinitiality of an ordered set is the cofinality of this set under the reversed ordering.

The cut (Λ^L, Λ^R) is called symmetric if $\kappa = \lambda$, and asymmetric otherwise. We call an ordered set $(S, <)$ (or ordered abelian group, or ordered field) symmetrically complete if every symmetric cut in S is principal. Note that the principal symmetric cuts are precisely the cuts with cofinality $(1, 1)$. Therefore, in dense linear orderings (and hence in ordered fields) there are no principal symmetric cuts. Consequently, a dense linear ordering is symmetrically complete if and only if all of its cuts are asymmetric.

For example, $\mathbb{Z}$ and $\mathbb{R}$ are symmetrically complete, but $\mathbb{Q}$ is not. In $\mathbb{Z}$ and $\mathbb{R}$, every cut is principal; in $\mathbb{Z}$ all of them have cofinality $(1,1)$, and in $\mathbb{R}$ they have cofinalities $(1,\aleph_0)$ or $(\aleph_0,1)$. In contrast, in $\mathbb{Q}$ the cuts have cofinalities $(1,\aleph_0)$, $(\aleph_0,1)$ and $(\aleph_0,\aleph_0)$.

Symmetrically complete ordered fields

The following theorem is a special case of a theorem proven in [KK 2015] for ordered abelian groups:

Theorem

The Banach Fixed Point Theorem holds in every ordered field whose order ball space is spherically complete.

At an early point during the preparation of [KK 2015], we proved the following result; only later we learned that it had been proven already in [S 2004]:

Theorem

An ordered field is symmetrically complete if and only if its order ball space is spherically complete.

Symmetrically complete ordered fields

Together, these two theorems imply:

Theorem ([KKS 2015])

The Banach Fixed Point Theorem holds in every symmetrically complete ordered field.

Existence?

But are there any symmetrically complete ordered fields other than $\mathbb{R}$? If you have a problem of a combinatorial nature that you cannot solve yourself, ask Saharon Shelah - and so we did. His answer: "I already proved that in 2004."

Theorem ([S 2004])

Every ordered field can be embedded in a symmetrically complete real closed field.

Characterization? Construction?

Shelah's paper did not answer these questions: How can these fields be characterized and can they be constructed as power series fields? The following results have been proven in [KKS 2015]:

Theorem

An ordered field is symmetrically complete if and only if it is of the form $\mathbb{R}((\Gamma))$ with Γ a dense strongly symmetrically complete ordered abelian group. Every such Γ is divisible, so $\mathbb{R}((\Gamma))$ is a nonstandard model of $\mathbb{R}$.

An ordered set is dense if for every two elements $a < b$ there is c such that $a < c < b$. A dense ordered set is strongly symmetrically complete if every cut has cofinality (κ, λ) with $\kappa \neq \lambda$ and at least one of κ , λ uncountable.

Hahn products

Now we have the same problems as before, but for ordered abelian groups. For them, we need the analogue of power series fields.

Given a linearly ordered index set I and for every $i \in I$ an arbitrary abelian group C_i , we define a group called the Hahn product, denoted by $\mathbf{H}_{i\in I}\, C_i$. Consider the product $\prod_{i\in I} C_i$ and an element $c = (c_i)_{i\in I}$ of this group. Then the support of c is the set $\operatorname{supp}(c) := \{i \in I \mid c_i \neq 0\}$. As a set, the Hahn product is the subset of $\prod_{i\in I} C_i$ containing all elements whose support is a wellordered subset of I. The Hahn product is a subgroup of the product group. Indeed, the support of the sum of two elements is contained in the union of their supports, and the union of two wellordered sets is again wellordered.
If all C_i are ordered, then an ordering on $\mathbf{H}_{i\in I}\, C_i$ can be defined just as it is done for power series fields.

Strongly symmetrically complete ordered abelian groups

Theorem

A nontrivial densely ordered abelian group $(G, <)$ is strongly symmetrically complete if and only if it is a Hahn product, with all C_i isomorphic to $\mathbb{R}$ and an extremely symmetrically complete index set I.

A dense ordered set is extremely symmetrically complete if it is strongly symmetrically complete and in addition, its coinitiality and cofinality are both uncountable.

Through the previous theorems, we have reduced the construction problem to the lowest level: that of ordered sets. While the proofs of those theorems is already tough enough, the proof of the following result caused us even more problems.

Theorem

Every ordered set can be embedded in some extremely symmetrically complete ordered set.

So far we have no characterization of symmetrically complete ordered sets; they do not admit straightforward analogues to power series or Hahn product constructions. This could be a subject for further research.

Working our way up

Recall our theorem on strongly symmetrically complete ordered abelian groups:

Theorem

A nontrivial densely ordered abelian group $(G, <)$ is strongly symmetrically complete if and only if it is a Hahn product, with all C_i isomorphic to $\mathbb{R}$ and an extremely symmetrically complete index set I.

Because of their canonical form, depending only on the index set I, we obtain as a corollary of the previous theorem:

Theorem

Every ordered abelian group can be embedded in a symmetrically complete ordered abelian group.

Working our way up

Recall our theorem on symmetrically complete ordered fields:

Theorem

An ordered field is symmetrically complete if and only if it is of the form $\mathbb{R}((\Gamma))$ with Γ a dense strongly symmetrically complete ordered abelian group. Every such Γ is divisible, so $\mathbb{R}((\Gamma))$ is a nonstandard model of $\mathbb{R}$.

Because of their canonical form, depending only on Γ, we obtain an alternative proof of Shelah's theorem, as a corollary of the previous theorem:

Theorem

Every ordered field can be embedded in a symmetrically complete real closed field.

Hausdorff

Already in the years 1906-1908, Hausdorff ([Hf 1908]) constructed ordered sets with prescribed cofinalities for all of its cuts. However, he did not discuss how to embed arbitrary ordered sets in such special ordered sets. Moreover, the constructions we present in [KKS 2015] lead more directly to the ordered sets we need.

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More detailed information

Preprints and further information:

The Valuation Theory Home Page
http://math.usask.ca/fvk/Valth.html.