The Axiom of Choice and large cardinals

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- Axioms of ZF set theory:
  - (Extensionality)
  - (Foundation)
  - Pairing
  - Union
  - Power set
  - Separation Scheme
  - Collection Scheme
  - Infinity
- Mostly closure properties, intuitively "constructive".
- (+ Infinity)

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– Standard full set theory: ZFC = ZF + Axiom of Choice (AC).

– AC is intuitively “non-constructive”...

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  etc...

- "Non-constructive": c not specified via any definition.

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Claim: Picture under ZF(C) + Large cardinal axioms suggests otherwise

(LCs: Increasing hierarchy of principles, artificially interrupted by AC)

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$V =$ the universe of all sets.

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ZF proves reflection: If a sentence φ is true (in V) then φ is true in some V_α.

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Large cardinal axioms strengthen reflection/inaccessibility.

Cardinal $\kappa > \omega$ with properties (i) and (ii) is inaccessible (ZFC).

Many large cardinal (LC) notions have been isolated, e.g. (in increasing consistency strength):

ZF / ZFC

Inaccessible cardinals

Mahlo

Reflecting

Weakly compact

Ramsey

Measurable

Strong

Woodin

Supercompact

Huge

Rank to rank

Also, many more intermediate levels.

Consistency strength

Definition 0.2.

Say $T \leq_{\mathrm{Con}} U$ iff PA proves

$$\text{if } U \text{ is consistent, so is } T.$$

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- $\leq_{\mathrm{Con}}$ gives partial order of theories.
- LCs essentially linearly ordered by consistency strength.
- Theories of form ZFC + LCs provide measure of consistency strength of other theories of form ZF + φ.

Elementary embeddings

Definition 0.3.

If M, N are structures for a first-order language $\mathcal{L}$, an elementary embedding

$$j : M \to N$$

is a function j preserving truth:

$$M \models \varphi(\vec{x}) \iff N \models \varphi(j(\vec{x}))$$

for formulas φ of $\mathcal{L}$ and $\vec{x} \in M^{<\omega}$.

(Default $\mathcal{L} = \text{LST} = \{\in, =\}$.)

Elementary embeddings and DC

Definition 0.4.

Dependent choice (DC): For every binary relation R over a set $X \neq \emptyset$, if

$$\forall x \in X \, \exists y \in X \, [xRy]$$

then $\exists \langle x_n \rangle_{n<\omega} \subseteq X$ such that

$$x_n R x_{n+1}$$

for all $n < \omega$.

Definition 0.5.

(ω, ω)-downward Lowenheim-Skoelm asserts that for every countable first-order language and every $\mathscr{L}$-structure N, there is a countable M and an elementary embedding $j : M \to N$.

Fact 0.1.

(ZF) DC is equivalent to (ω, ω)-downward Lowenheim-Skoelm.

Compactness and AC

Definition 0.6.

First order compactness asserts the compactness theorem for first order logic (for all first order languages).

Fact 0.2.

(ZF) AC is equivalent to first order compactness.

Large cardinals $\geq$ measurable typically exhibited by elementary

$$j : V \to M$$

where V = universe of all sets, and $M \subseteq V$ is a transitive class.

Transitive: For all $x \in M$, have $x \subseteq M$.

Critical point of j is least ordinal κ with $j(\kappa) > \kappa$.

Measurable cardinal is critical point of some j (in ZFC).

Many large cardinals have characterizations in terms of:

- elementary embeddings,
- ultrafilter existence,
- compactness principles.

E.g. κ is strongly compact iff for every theory $T \subseteq \mathbb{L}_{\kappa,\kappa}$, if every subset of T of size $< \kappa$ is satisfiable, then T is satisfiable.

Inner models

Gödel's constructible universe L

- Contains sets constructed in an explicit, definable manner.
- ZFC holds in L, with AC definably, uniformly.
- L has hierarchy $\langle L_\alpha \rangle_{\alpha \in \mathrm{OR}}$ with nice properties, e.g. condensation.
- L admits small large cardinals, e.g. weakly compact

Theorem 0.7 (Scott).
If there is a measurable cardinal then $V \neq L$.

- Can see "$V = L$" as strong version of AC.
- Conclusion: LCs disprove strong AC.

Inner model theory

- Goal: find *L*-like models *M* with larger cardinals, preferably “canonical”.
- *M* built like *L* but feeding in canonical LC information.
- Successful through many Woodin cardinals.
- Extensive structure
- Intuitive evidence for validity of LCs
- Woodins give *L*-like inner models with Woodins.
- Supercompact? There are problems...

Determinacy

Two player games: Fix a set

$$A \subseteq {}^{\omega}\omega = \{f \mid f : \omega \to \omega\}.$$

The game G_A:

- Two players, 1 and 2.
- The players alternate playing integers, Player 1 playing first x_0, then Player 2 playing x_1, etc, producing sequence

$$x_0, x_1, x_2, \ldots$$

- Player 1 wins iff $(x_0, x_1, x_2, \ldots) \in A$.

A winning strategy (w.s.) for Player $i = 1, 2$:

- a function σ on set of partial plays, specifying next moves,
- Player i always wins if he/she follows σ at every step.

Say G_A is determined if some player has a w.s.

Axiom of Determinacy (AD) says that all games G_A are determined.
Projective Determinacy (PD) says that G_A is determined for all projective sets A.

Theorem 0.8 (Martin).

Assume ZFC.

(i) G_A is determined for all Borel sets A.

(ii) If there is a measurable cardinal then likewise for analytic A.

- Under ZFC, AD is false.
- Determinacy seen as appropriate for “non-pathological” sets A.

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- Under ZFC, AD is false.
- Determinacy seen as appropriate for "non-pathological" sets *A*.
- ZF + AD + choice fragments extensively studied. Consequences:
  - Lebesgue measurability,
  - property of Baire, etc,
  - inner models of ZFC with large cardinals,
  - aspects of choice (e.g. $AC_{\omega,\mathbb{R}}$)
  - many ultrafilters

Theorem 0.9 (Martin, Harrington, Neeman, Woodin).

(ZFC) PD holds iff for each $n < \omega$ there is a canonical proper class inner model of ZFC with n Woodin cardinals.

Although AD contradicts AC:

Theorem 0.10.

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Suppose $B \subseteq \mathbb{R} \times \mathbb{R}$. A uniformization of B is a function c with domain

$$x \in \mathrm{dom}(c) \iff \exists y[(x, y) \in B],$$

and $B(x, c(x))$ for all $x \in \mathrm{dom}(c)$.

Given B of certain complexity, is there a uniformization? Of the same complexity?

ZF + AD proves there is in many instances (not all!).

Stronger large cardinals

– Stronger LCs via $j : V \to M$ with $M \approx V$.
– Reinhardt: $j : V \to V$.

Theorem 0.11 (Kunen).

There is no elementary $j : V \to V$.
There is no λ and elementary $k : V_{\lambda+2} \to V_{\lambda+2}$.

Proof steps:

(i) If $j : V \to V$ elementary, get λ with $j(\lambda) = \lambda$, hence

$$k : V_{\lambda+2} \to V_{\lambda+2}$$

by restricting j.

(ii) **Using AC**, combinatorics: There is no such k.

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Thus...

– Work in ZF. Investigate large cardinals $k : V_{\lambda+2} \to V_{\lambda+2}$ and beyond.
– Much structure...

Theorem 0.12 (Suzuki, 1999).

Assume ZF. Then there is no $j : V \to V$ which is definable from parameters.

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Adapting this proof...

Theorem 0.13 (S.).

Assume ZF and $j : V_\lambda \to V_\lambda$ is Σ_1-elementary where λ is a limit ordinal. Then j is not definable from parameters over V_λ.

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Remark 0.14 (Folklore).

Assume ZF and $j : V_{\lambda+1} \to V_{\lambda+1}$ is elementary, where λ is a limit ordinal. Then j <u>is</u> definable from a parameter in $V_{\lambda+1}$.

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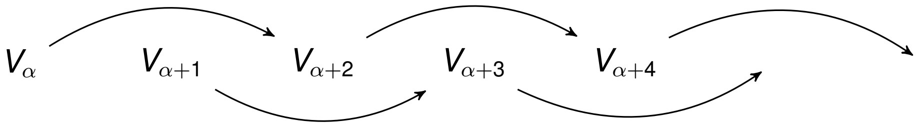
Theorem 0.15 (S.).

Assume ZF and $j : V_{\alpha+1} \to V_{\alpha+1}$ is elementary. Then j is not definable over $V_{\alpha+1}$ from any parameter in V_α.

Theorem 0.16 (Goldberg, S.).

Suppose $j : V_\alpha \to V_\alpha$ is elementary. Then j is definable over V_α from parameteters iff α is an odd ordinal.

– If $j : V \to V$, then for eventually all α, there is $k : V_\alpha \to V_\alpha$.
– So the cumulative hierarchy is eventually periodic in structure:



– Recall $V_{\gamma+1} = \mathcal{P}(V_\gamma)$ uniformly
– Goldberg has found further distinctions between odd/even, and AC-like combinatorial principles follow from $j : V \to V$.

(Recall AD proves existence of many optimal choice functions...)

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Choice functions mod measure one.

Let $B \subseteq V_\alpha \times V_\alpha$.

A measure one uniformization of B is a function c with $\mathrm{dom}(c) \subseteq V_\alpha$ and:

– $x \in \mathrm{dom}(c) \implies B(x, c(x))$, and
– $\mathrm{dom}(c)$ is of "measure one".

$j : V \to V$ proves, in many instances, the existence of measure one uniformizations.

Sharps

Sharps represent elementary embeddings $j : M \to M$, seen externally (unlike Reinhardt embeddings).

- $0^{\#}$ represents $j : L \to L$.
- $X^{\#}$ represents $j : L(X) \to L(X)$ such that $j \restriction X \cup \{X\} = \mathrm{id}$.

Theorem 0.17 (Goldberg).

If $j : V \to V$ then $X^{\#}$ exists for every set X.

And (more generally):

Theorem 0.18 (S.).

If $j : V \to V$ then $M_n^{\#}(X)$ exists for every set X and every integer n.

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And (more generally):

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If $j : V \to V$ then $M_n^{\#}(X)$ exists for every set X and every integer n.

- $M_n(X)$ is canonical inner model containing X and n Woodin cardinals above.
- $M_0^{\#}(X)$ is equivalent to $X^{\#}$.

Corollary 0.19.

If $j : V \to V$ then PD holds in V and every set-generic extension.

This should go much further than PD.

Under $j : V \to V$, $X^\#$ exists, so V is far from $L(X)$. Further:

Theorem 0.20 (S.).

If $j : V \to V$ then there is no set X such that $V = \mathrm{HOD}(X)$.

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Theorem 0.21 (Goldberg, Usuba).

If $j : V \to V$ then no set-forcing extension of V models AC.

Consistency

Woodin's $I_{0,\lambda}$ says "there is an elementary $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ with $\mathrm{cr}(j) < \lambda$".

Let T be the theory ZFC + $I_{0,\lambda}$.
Let T^+ be T + "$V_{\lambda+1}^{\#}$ exists".

– T, T^+ extensively studied by Woodin and others.
– Many analogues between $L(V_{\lambda+1})$ under T and $L(\mathbb{R})$ under ZF $+$ AD.

Theorem 0.22 (S.).

If T^+ is consistent, then so is ZF + "$k : V_{\lambda+2} \to V_{\lambda+2}$ is elementary".

$\Longrightarrow$ If ZF disproves $j : V \to V$, proof significantly different from Kunen's.

Goldberg observed that T^+ can be reduced to T.

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This isn't even the top... Super-Reinhardt, total Reinhardt, Berkeley cardinals deeply transcend Reinhardt.

Berkeley cardinals also violate AC in a stronger manner.

Are they consistent?

Inner models and supercompacts

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Different form? Many branches simultaneously?

Amenability a strong form of AC, falsified by supercompacts.

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Consistency strength hierarchy

- Theories of form ZFC + LCs provide standard yardstick.
- Are AC-LCs actually cofinal in consistency strength order?
- (If not, not a sufficient yardstick.)

Remarks/Speculations:

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- Full AC formulated a century ago, prior to LCs, determinacy
- LCs and determinacy exhibit detailed structure, cohesive picture
- LCs intuitively extend other "constructive" axioms of ZF
- We don't seem to have natural candidates for AC-LCs cofinal with non-choice LCS.
- Do LCs really stop at $V_{\lambda+2}$?
- My expectation: AC is false, LCs are the correct organizing principle.

Possible picture of V:

- ZF holds
- $j : V \to V$ is elementary,
- $V_\lambda \models$ ZFC where $\lambda = \lambda_j$,
- choice fails in $V_{\lambda+2}$ (and above).
- By assuming AC, we could be limiting our view to V_λ, ignoring upper universe.

Thank you for listening!

References:

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- Extenders under ZF and constructibility of rank-to-rank embeddings, Schlutzenberg (see arXiv).