How geometry became structural

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Motivation of this talk

Nineteenth-century geometry is characterized by a "structural turn" of the field:

... a shift from the traditional understanding of geometry as the science of extension to that as a science of abstract structures.

... a new, model-theoretic understanding of geometrical theories.

How did this transition go?

The "structural turn" in geometry

Concerning geometry, there was a gradual transformation from the study of absolute or perceived space – matter and extension – to the study of structures. (Hellman & Shapiro 2019, p.8)

A striking and characteristic feature of the history of geometry during the nineteenth century is the increasing "abstracteness" of its language and the progressive formalization of its procedure. (Nagel 1939, p.217)

Structuralist methodology

The new conception of geometry is not just a consequence of emergence of non-Euclidean geometry, i.e. of the "problem of the multiplicity of geometries" (Coffa 1986).

More importantly, it is a consequence of several innovations in the methodology of geometry, developed in different subfields at the time.

The new "structural methods" led to an abstraction from the nature of primitive spatial objects and thus from the traditional subject matter of geometry. (see Reck (2000, 2003))

Aims in this paper

Focus on the development of two related structural methods in nineteenth-century geometry:

1. Transfer principles in analytic projective geometry and in Klein's Erlangen Program (1872): mappings between different systems that show the equivalence of geometries.
2. Hilbert's use of model constructions and coordinatization in *Grundlagen der Geometrie* (1899), e.g., in his independence and consistency proofs.

Transfer principles in projective geometry

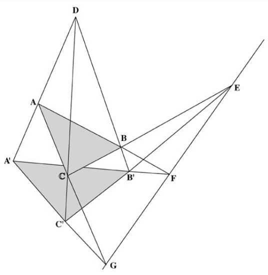
The principle of duality in PG

Principle of duality

For any theorem of plane projective geometry we get another theorem of plane projective geometry by interchanging (1) the basic terms ‘point’ and ‘line’, (2) the basic relations ‘lies on a line’ and ‘goes through a point’ and (3) accordingly, all notions defined from these basic notions.

Desargues' theorem

Theorem	**Dual theorem**
If three lines joining the corresponding vertices of two triangles ABC and $A^{'}B^{'}C^{'}$ meet in a single point, then the three intersections of pairs of corresponding sides G, F, E lie on a straight line.	If the intersections of corresponding sides of ABC and $A^{'}B^{'}C^{'}$ all lie on a straight line, then the lines joining corresponding vertices meet in a single point.



The axiomatic approach duality

- Moritz Pasch formulated the first axiom system for projective geometry in *Vorlesungen über Neuere Geometrie* (1882).
- §12 discusses *reciprocity* (i.e. duality) as a property of statements of projective geometry.
- The justification of the principle of duality in space is based on (i) the axiomatic presentation of the theory and (ii) the "rigorous deductive method": all theorems of projective geometry are provable from the given set of axioms.

see Schlimm (2010)

Pasch on reciprocity

The law of reciprocity can be verified first for the graphical sentences of §§7, 8, 9, since the reciprocal sentence of every sentence also belongs to this group.

Every other sentence to be considered here is a consequence from these sentences. (...) Every theorem is thus the result of a consideration in which only graphical base concepts are mentioned and in which one only refers to the graphical sentences mentioned above.

If one substitutes systematically the word "point" by "plane", "plane" by "point" and the used theorems by its reciprocals in this approach, then its correctness remains untouched; but as a result one finds "point" and "plane" interchanged, i.e. one has proved the reciprocal theorem. (Pasch 1882, p.96)

(Plane) Duality reconstructed

Definition (Projective plane)

A projective plane is an incidence structure (of the form $\langle \mathfrak{p}, \mathfrak{L}, I \rangle$) with the following properties:

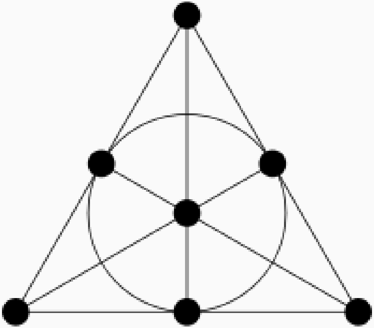
(PP1) Every two points are incident with a unique line.

(PP2) Every two lines are incident with a unique point.

(PP3) There exist at least four points, three of which are not collinear.

(PP4) There exist at least four lines, three of which are not concurrent.

Fano plane



An axiomatic (or proof-theoretic) account of duality

- Plane PG can be formulated in a language $\mathcal{L}_{PG}$ with two one-place predicates P, L (for 'point' and 'line') and a two-place predicate I (for 'incidence' between points and lines).
- Let φ^d be the *dual* statement of φ obtained by translation $(.)^d : \mathcal{L}_{PG} \rightarrow \mathcal{L}_{PG}$ st.
  (i) $[P(x)]^d = L(x)$
  (ii) $[L(x)]^d = P(x)$
  (iii) $[I(x, y)]^d = I^*(x, y)$

Explication (Principle of Duality)

For each statement $\varphi \in \mathcal{L}_{PG}$: if $\mathbb{P} \vdash \varphi$, then $\mathbb{P} \vdash \varphi^d$.

Eder & Schiemer (2018)

The analytic approach to duality

A justification of duality in work by Julius Plücker (1801-1868), based on the analytic representation of geometric concepts:

E.g. linear equation $y + xu + v = 0$ presents a straight line in the plane.

... on its usual interpretation, u, v are treated as constants that determine a collection of points on a line.

... one can interpret u, v as "line coordinates". If x, y treated as constants and u, v as variables, then equation determines a collection of lines (or a line curve).

Plücker on reciprocity

Plücker in *System der Geometrie des Raumes* (1846):

> *Every geometrical relation is to be viewed as the pictorial representation of an analytic relation, which, irrespective of every interpretation, has its independent validity. Consequently, the principle of reciprocity properly belongs to analysis, and only because we are accustomed to (...) express the matter in geometrical language, does it seems to be an exclusively geometrical principle. (ibid, p.322)*

Duality as a transfer principle

The term ‘transfer principle’ first occurs in Plücker’s work in the context his discussion of reciprocity.

System der analytischen Geometrie (1935): three “main classes of transfer principles”, each based on the interpretation of analytic equations in different coordinate systems: (i) mappings between point coordinate systems, (ii) mappings between line coordinate systems; (ii) the reciprocity of line and point coordinate systems.

Plücker argues that, due to the principle of linear reciprocity, “one can transfer the relations of one of two reciprocal systems to the other one” (ibid., p.48).

Hesse's transfer principle

Transfer principles in projective geometry as a generalization of the principle of duality.

L. O. Hesse, "Über ein Übertragungsprinzip" (1866):

... a 1-1 correspondence $\Phi : P \to p$ between points $P = (x, y)$ of the complex projective plane and pairs of points $p = \{\lambda_1, \lambda_2\}$ on the complex projective line,

... given by a quadratic function of the form:

$$A\lambda^2 + B\lambda + C = 0$$

with A, B, C linear functions of coordinates x, y of P.

A relation between geometries

If one makes to correspond in a univocal way to each point in the plane a pair of points on the straight line and vice versa, to each pair of points a point in the plane, one has a transfer principle which reduces the geometry of the plane to the geometry of the straight line and vice versa. (Hesse 1866, p.15)

Hesse's principle

Hesse's mapping Φ is structure-preserving, i.e. it preserves "relations between figures" ("Figurenverhältnisse") in the two systems.

This is established by Hesse in terms of a number of "fundamental theorems" ("Fundamentalsätze") that show how central or primitive projective properties of the objects in the first system correspond to properties of pairs of points on the fundamental line.

Each theorem about configurations in the plane can be translated into a theorem about configurations on projective line.

Transfer principles in Klein's Erlangen Program

"*Comparative considerations of recent developments in geometry*" (1872) is Felix Klein's programmatic paper related to his appointment at the university of Erlangen.

Main idea: to study geometries in terms of their transformation groups and related invariants.

Geometry as an invariant theory

Given a manifold and a group of transformations of the same; to investigate the configurations belonging to the manifold with regard to such properties as are not altered by the transformations of the group.

Given a manifold and a group of transformations of the same; to develop the theory of invariants relating to that group.

Transfer principles and the equivalence of geometries

Klein's focus in (1872) is not (primarily) on particular geometries but on the *comparison* of different theories in terms of their transformation groups.

In particular, he uses "transfer principles" ("Übertragungsprinzipien") to identify geometries with different domains.

Roughly put, two geometries with isomorphic transformation groups are "essentially similar".

"Transfer by mapping"

Suppose a manifoldness A has been investigated with reference to a group B. If, by any transformation whatever, A be then converted into a second manifoldness A', the group B of transformations, which transformed A into itself, will become a group B', whose transformations are performed upon A'. It is then a self-evident principle that the method of treating A with reference to B at once furnishes the method of treating A' with reference to B', i.e., every property of a configuration contained in A obtained by means of the group B furnishes a property of the corresponding configuration in A' to be obtained by the group B'.

Definition (Equivalent geometries)

Two geometries (M, G) *and* (M', G') *are equivalent iff there exists a bijection* $F : M \to M'$ *and a group isomorphism* $\alpha : G \to G'$ *induced by* F *such that*

$$\text{for all } x \in M \text{ and for all } g \in G : \; F(g(x)) = (\alpha(g))(F(x))$$

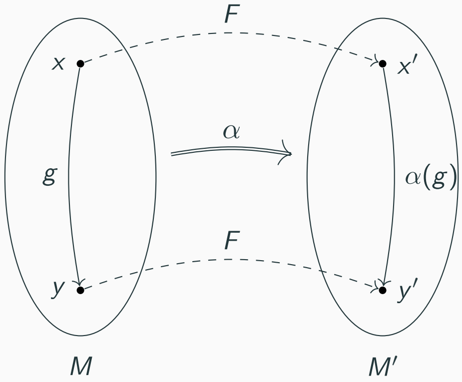


Figure 1: A transfer principle between manifolds M and M'.

Indifference to ontology

Klein in §5 titled "On the arbitrariness in the choice of the space-element":

> *As element of the straight line, of the plane, of space, or of any manifoldness to be investigated, we may use instead of the point any configuration contained in the manifoldness, - a group of points, a curve or surface, etc. (...) But so long as we base our geometrical investigation on the same group of transformations, the geometrical content [Inhalt der Geometrie] remains unchanged. That is, every theorem resulting from one choice of space element will also be a theorem under any other choice; only the arrangement and correlation of the theorems will be changed.*

Klein's 'methodological structuralism'

A geometry is determined not by "the particular nature of the elements of the manifold on which it is defined" but by the structure encoded in its transformation group. (cf. Torretti 1978, Marquis 2008)

Two geometries defined on different manifolds are structurally equivalent if there exists a transfer principle between them, i.e. if they are characterized by the same transformation group (up to isomorphism).

Modern axiomatics and modeling

Hilbert's axiomatics

Hilbert's *Grundlagen der Geometrie* (1899):

- Axiom systems implicitly define the primitive terms and can be interpreted relative to different models.
- *Metatheoretic* results concerning the consistency of axiom systems, independence of particular axioms, embeddability results concerning models of different (sub-)theories.
- These results are presented in a *model-theoretic* way, i.e. by the construction of models with the relevant geometrical properties.

Hilbert's 'way of understanding' the independence results therefore introduces, and is based on, the distinction between the axiomatized theory on the one hand and the various models on the other. (Hallett 2010, p.453)

There is no doubt that Hilbert's Foundations of Geometry was one of the main gateways of model-theoretical thinking into twentieth-century logic and philosophy. (Hintikka 1988, p.6)

Question

How did Hilbert understand the method of (re-)interpretation in his independence/consistency proofs in 1899? Are these proofs really model-theoretic in character?

$\Rightarrow$ Eder, G. & Schiemer, G. “Hilbert, duality, and the geometrical roots of model theory”, RSL, (2018)

Hilbert's axiomatization of Euclidean geometry

Hilbert presents an axiom system for *Euclidean* geometry with six primitive terms: 'point', 'line', 'plane', 'between', 'lies on', 'congruence'.

Five *groups* of axioms:

1. Axioms of connection (the 'projective basis' of his system):
   Ax 1-2 concern planar geometry, Ax 3-7 solid geometry
2. Axioms of order
3. Axioms of congruence
4. Axiom of parallels
5. Axioms of continuity (Archimedean axiom & axiom of completeness)

Two types of interpretations in *Grundlagen*

1. Hilbert's C& I proofs are based on different interpretations of subsystems of Euclidean geometry in field theories (i.e. Pythagorean, non-Archimedean, complete, etc.): models as interpretations of geometry in field theory.
2. In algebra of segments, interpretations of field theory into geometry in order"to enable the use of algebraic methods in proving geometrical theorems via coordinatization." (Baldwin 2018)

Consistency (‘compatibility’) of the AS

The axioms of the five axiom groups given in chapter 1 do not stand in contradiction with each other, i.e. it is no possible to deduce from them via logical inferences a fact which contradicts one of the given axioms. To see this, it suffices to present a geometry in which all axioms of the five groups are satisfied. (Hilbert 1902, §9)

A model construction

Hilbert's proof of the consistency of the plane axioms (without the axiom of completeness) in §9:

- A set ("*Bereich*") Ω of algebraic numbers containing 1 and closed under operations $+, -, \times, \div$ and $\sqrt{1+x^2}$.
- Points identified with tuples (x, y) of numbers in Ω, lines with ratios $(u : v : w)$ of numbers in Ω.
- A point (x, y) lies on a line $(u : v : w)$ if $ux + vy + w = 0$ holds.

Hilbert concludes:

> *(...) given this, as one can easily see, axioms I 1-3 and IV are satisfied. (ibid, §9)*

Relative consistency

Hilbert is emphasizing of the relative character of his consistency and independence proofs:

> *Any contradiction in the consequences from our axioms I-IV, V 1 would thus also have to be recognisable in the arithmetic of the domain Ω.*

Thus, what's important for Hilbert is "the relationship *between* mathematical theories" (Hallett 2010). His consistency and independence proofs are dependent on a background theory based on which the geometrical models are constructed.

Hilbert "Über den Zahlbegriff" (1900)

- Hilbert's background or "base" theory for the metatheoretic study of his geometrical AS is the theory of complete ordered fields specified in Hilbert 1900a ("Über den Zahlbegriff").
- Primitive terms: *Zahl*, $+, \times, <, 1$ (neutral element); the axiom system consists of
  - (i) Axioms of connection
  - (ii) Axioms of calculation
  - (iii) Axioms of order
  - (iv) Axioms of continuity (Archimedean, Completeness).

An explication of Hilbert's method of interpretation

Definition

A translation f of L_S in L_T consists of L_T-formulas $\delta(x), \varphi_{R_i}(x_1, ...x_n)$ (for all primitive predicates $R_1, ...R_k$ of L_S), such that

1. $R_i(x_1, ...x_n)^f = \varphi_{R_i}(x_1, ...x_n)$
2. $(x = y)^f = (x = y)$
3. $(\neg\varphi)^f = \neg\varphi^f$
4. $(\varphi \wedge \psi)^f = \varphi^f \wedge \psi^f$
5. $(\forall x\varphi)^f = \forall x(\delta(x) \to \varphi^f)$

Interpretability

Definition

f is an interpretation of S in T iff.

1. *f is a translation of L_S in L_T*
2. *For every L_S-sentence φ: if $S \vdash \varphi$, then $T \vdash \varphi^f$*

Definition

S is interpretable in T iff. there is a interpretation of S in T.

Reconstructing Hilbert's proof

Hilbert's proofs of relative consistency can be made precise by means of the notion of *interpretability* between theories.

Hilbert's "model construction" can be understood in terms of a translation of the language of Euclidean planes into the language of Pythagorean fields.

It is then shown that this translation preserves theoremhood, i.e. any translation of a Euclidean theorem is provable from the theory of Pythagorean fields.

see, e.g., Baldwin (2018)

Hilbert & Bernays on the“method of arithmetization”

Hilbert & Bernays, *Grundlagen der Mathematik*, Vol 1 (1934):

> *With respect to the previous discussion of this problem [the consistency of an AS], this is addressed both in geometry and in the physical sciences through the method of arithmetization:*
>
> *One represents the objects of a theory by numbers and number systems and the primitive relations by equations and disequations such that the axioms of the theory are transformed either in arithmetical identities or in provable sentences on the basis of these translations, as is the case in geometry. (...)*
>
> *In this procedure arithmetic, i.e. the theory of real numbers (analysis) is presupposed as valid. (p.3)*

Summary: structuralism and model theory

The development of structuralism is often identified with emergence of model theory, more precisely, a model theoretic conception of theories (e.g. Scanlan 1988, Shapiro 1997)

However, the crucial model-theoretic notion in nineteenth-century geometry (including Hilbert) is *not* the interpretation of a theory in structures but the interpretation between theories.

Interpretations in the latter sense present a method to encode one theory in another theory, thus showing their shared structure.

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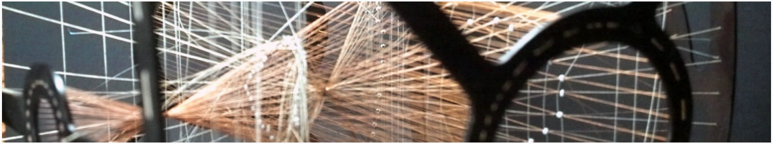

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