The exact consistency strength of “AD^+ + all sets are universally Baire”

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We will discuss the deep connection between determinacy and inner models with large cardinals.

- Games of length ω

finitely many Woodin cards

- Longer games and games on reals

& infinitely many Woodins

- Determinacy when all sets are universally Baire

cardinal λ that is a limit of Woodin cards & a limit of strong cardinals

Games in set theory

$A \subseteq {}^{\omega}\omega$.

I	n_0		n_2		$\dots$
II		n_1		n_3	$\dots$

I wins iff $(n_0, n_1, \dots) \in A$, o/w II wins.

A winning strategy for I is a function that dictates I what to play next depending on the previous moves in the game.

Def: A set of reals A is determined iff one of the players has a winning strategy in the game with payoff A.

What is determinacy good for?

Theorem (Mycielski, Swierczkowski, Mazur, Davis, 1960's)

If all sets of reals are determined, then all sets of reals

- *are Lebesgue measurable,*
- *have the Baire property, and*
- *have the perfect set property.*

Theorem ("The Wadge Brigade", Carroy-Medini-M, JML 2020)

If all sets of reals are determined and X is a zero-dimensional homogeneous space that is not locally compact, then X is strongly homogeneous.

All of these results have *local versions*.

Which games are determined?

consistency strength

- Gale-Stewart, 1953: Assume ZFC. Then every open and every closed set is determined.
- Martin, 1975: Assume ZFC. Then every Borel set of reals is determined.
- Martin, 1970: Assume ZFC and that there is a measurable cardinal. Then every analytic set is determined.
- Martin-Steel, 1985: Assume ZFC and there are n Woodin cardinals with a measurable cardinal above them all. Then every $\mathbf{\Sigma}^1_{n+1}$ set is determined.
- Gale-Stewart, 1953: Assuming AC there is a set of reals which is not determined.

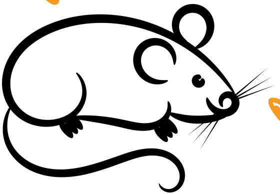
Determinacy and large cardinals

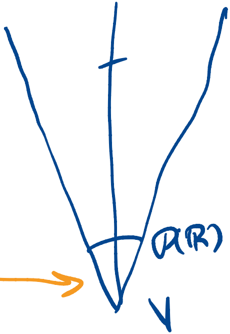
Are large cardinals necessary for the determinacy of these sets of reals?

In some sense ...

How can these large cardinals affect what happens with the sets of reals?

"mice" / little objects with large cardinals





large cardinals

?

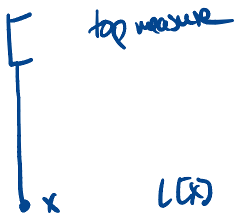
determinacy

An equivalence for Analytic Determinacy

Theorem (Harrington, Martin)

The following are equivalent.

1. *All analytic games are determined.*
2. $x^{\#}$ *exists for all reals* x. "the minimal nice model for a meas. card containg x"

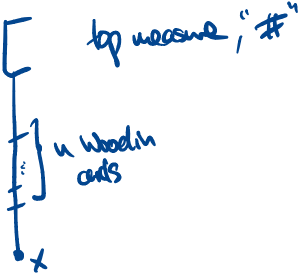


An equivalence for Projective Determinacy

Theorem (Neeman, Woodin)

Let $n \geq 1$. Then the following are equivalent.

1. $\boldsymbol{\Sigma}^1_{n+1}$*-determinacy.*
2. *For every $x \in \mathbb{R}$ the ω_1-iterable countable model of set theory with n Woodin cardinals $M_n^\#(x)$ exists.*



For (1) $\Rightarrow$ (2) see (M-Schindler-Woodin) "Mice with Finitely many Woodin Cardinals from Optimal Determinacy Hypotheses", JML 2020.

For (2) $\Rightarrow$ (1) see (Neeman) "Optimal proofs of determinacy II", JML 2002.

Games of countable length $\alpha > \omega$

$A \subseteq {}^{\alpha}\omega$.

I	n_0		n_2		...	n_ω	...	...
II		n_1		n_3	...		$n_{\omega+1}$ ...	

Theorem (Neeman, 2004)

Let $\alpha > 1$ be a countable ordinal and suppose that there are $-1 + \alpha$ Woodin cardinals with a measurable cardinal above them all. Then $\mathrm{Det}_{\omega \cdot \alpha}(\mathbf{\Pi}^1_1)$ holds.

Theorem (Aguilera-M, JSL 2020)

Suppose $\mathrm{Det}_{\omega \cdot (\omega+1)}(\mathbf{\Pi}^1_1)$. Then there is a premouse with $\omega + 1$ Woodin cardinals.

Woodin

ω Woodins

Larger countable ordinals

Theorem (Aguilera-M, JSL 2020)

Let $n < \omega$ and suppose $\mathrm{Det}_{\omega\cdot(\omega+n)}(\mathbf{\Pi}^1_1)$. Then there is a premouse with $\omega + n$ Woodin cardinals.

Theorem (Trang, 2013, building on Woodin)

Let α be a countable ordinal and suppose $\mathrm{Det}_{\omega^{1+\alpha}}(\mathbf{\Pi}^1_1)$. Then there is a premouse with ω^α Woodin cardinals.

Theorem (M, 2019)

Let α be a countable ordinal and suppose $\mathrm{Det}_{\omega^{1+\alpha}}(\mathbf{\Pi}^1_{n+1})$. Then there is a premouse with $\omega^\alpha + n$ Woodin cardinals.

$\exists$ ω Woodins

$\exists$ ω Woodins

$M_1^\#(A) \models AD$

Games on reals

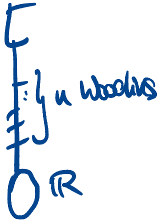
$A \subseteq {}^{\omega}({}^{\omega}\omega)$

I	$x_0 \in {}^{\omega}\omega$		x_2		...
II		x_1		x_3	...

Theorem (Aguilera-M, NDJFL 2020)

The following are equivalent:

1. *Projective determinacy for games on* $\mathbb{R}$;
2. $M_n^\sharp(\mathbb{R})$ *exists for all* $n \in \mathbb{N}$.



Another approach to strengthen determinacy

Keep playing games of length ω and impose additional structural properties on the model.

Suslin sets

Being Suslin is a generalization of analytic sets.

Definition

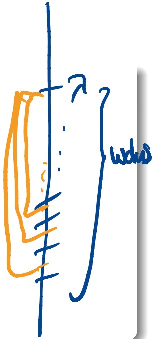
A set of reals is *Suslin* if it is the projection of a tree on $\omega \times \kappa$ for some $\kappa \in \mathrm{Ord}$.

$A = p[T]$

Under AC: Trivially, every set of reals is Suslin (for a tree on $\omega \times 2^{\aleph_0}$).

Under AD: There are natural models in which not every set of reals is Suslin, e.g. $L(\mathbb{R})$.

$\mathrm{AD}+$ all sets of reals are Suslin



Theorem (Woodin, Derived model construction, 1980's)

Suppose there is a cardinal λ that is

- *a limit of Woodin cardinals, and*
- *a limit of $<\lambda$-strong cardinals.*

Then there is a model of $\mathrm{AD}_{\mathbb{R}}$

$$\text{“}\mathrm{AD}+ \textit{all sets of reals are Suslin}\text{”}.$$

Theorem (Steel, 2008)

This is optimal.

The connection to $\mathrm{AD}_{\mathbb{R}}$

Theorem (Martin, Woodin, 1980's)

Under AD*, the following are equivalent.*

1. $\mathrm{AD}_{\mathbb{R}}$.
2. *All sets of reals are Suslin.*

${}^{\omega}\omega \; (= \mathbb{R})$

I	x_0	x_2	...
II	x_1	x_3	...

A further strengthening: universally Baire sets

Being universally Baire is a strengthening of being Suslin. $A = p[T]$

$A = p[S]$, S a very nice tree

Definition

Let (S, T) be trees on $\omega \times \kappa$ for some ordinal κ and let Z be any set. We say (S, T) *is Z-absolutely complementing* iff

easy $p[S] \cap p[T] = \emptyset$

$$p[S] = {}^{\omega}\omega \setminus p[T]$$

$p[S] \cup p[T] = {}^{\omega}\omega$

in every $\mathrm{Col}(\omega, Z)$-generic extension of V.

Definition (Feng-Magidor-Woodin)

A set of reals A is *universally Baire (uB)* if for every Z, there are Z-absolutely complementing trees (S, T) with $p[S] = A$.

Under AC: Not every set of reals is uB (as uB sets have regularity properties).

How strong is "AD^{+} + all sets are universally Baire"?

non-trivial generalization of Woodin's derived model construction

Theorem (Larson-Sargsyan-Wilson, 2014)

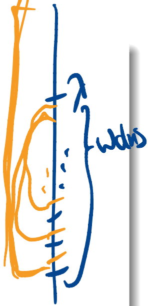
Suppose there is a cardinal λ that is

- *a limit of Woodin cardinals, and*
- *a limit of (fully) strong cardinals.*

Then there is a model of

$$\text{"}\mathrm{AD}^{+} + \textit{all sets of reals are universally Baire"}.$$

Note: This model cannot be of the form $L(\mathbb{R})$, $L(\mathcal{P}(\mathbb{R}))$, ...



Conjecture (Sargsyan)

This is optimal.

Sargsyan's conjecture holds

proper class

Theorem (M, 2021)

Suppose there is a model of

$$\text{“AD}^+ + \textit{all sets of reals are universally Baire”}.$$

Then there is a transitive model $\mathcal{M}$ of ZFC containing all ordinals such that $\mathcal{M}$ has a cardinal λ that is

- *a limit of Woodin cardinals, and*
- *a limit of (fully) strong cardinals.*

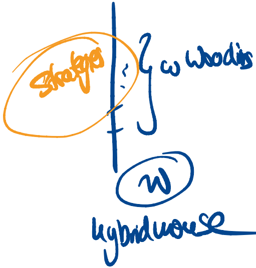
Some ideas behind the proof

Part I: HOD analysis - getting a translatable structure

Analyse the minimal model of

"$AD_{\mathbb{R}}$ + all sets are uB"

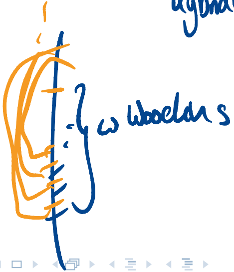
Consider direct limits of "hod pair"



Part II: The translation procedure

Translate strategies into

strong cardinals



The triple helix

The connection between determinacy and inner models with large cardinals gets a third component.

The triple helix

The connection between determinacy and inner models with large cardinals gets a third component.

translatable models

Hybrid mice

Part II

Part I

limit of Woodins & strongs

AD + all sets are uB

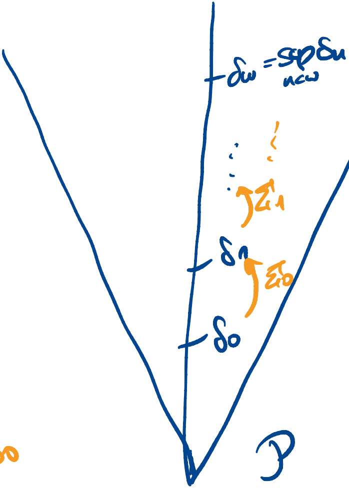
Determinacy

Mice with large cardinals

first part of the talk

What is a hybrid mouse?

What is a hybrid mouse? (Sargsyan)



δ_i Woodin cardinals

Σ_i iteration strategies (with nice properties)

Σ_i is an it. strategy for $\approx P|\delta_0$

Part I: HOD analysis - getting a translatable structure

$\mathcal{M}_\infty$ = direct limit of all nice hod pairs

$(\mathcal{P}, \Sigma)$

hybrid mouse

nice iteration strategy for $\mathcal{P}$

$\mathcal{M}_\infty$:

The fact that in our model all sets are uB allows to prove that $\mathcal{M}_\infty$ has an iteration strategy that

- can be interpreted onto generic extensions
- is guided by term relations (in these gen. extensions)

$x_{\tau_i} (i<\omega)$ $\quad i : \mathcal{M}_\infty | \delta_0 \xrightarrow[\Sigma_0]{} N \quad i(\tau_i^{\mathcal{M}_\infty}) = \tau_i^N$

Part II: The translation procedure

hybrid $\mathcal{W}$ $\longrightarrow$ ordinary $\mathcal{M}$

The following theorem extends work of Steel, Zhu, and Sargsyan.

ω Woodin cards + 1 strong

Theorem (M, 2021)

Let $\mathcal{W}$ be a translatable structure. Then the result of our translation procedure $\mathcal{M}$ inside $\mathcal{W}$ is a proper class model with a cardinal λ that is

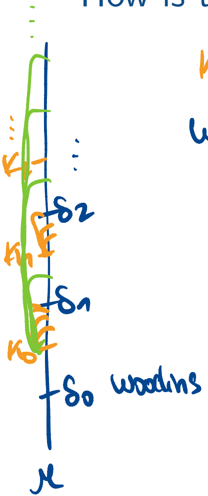
- *a limit of Woodin cardinals, and*
- *a limit of (fully) strong cardinals.*

Additional feature: All hulls of $\mathcal{M}$ are iterable.

Part II: The translation procedure

How is the translation procedure defined?

κ_i = the least $< \delta_{i+1}$-strong cardinal



We construct M "level by level":

- if there is a fully backgrd. extender in W, that we could add to the current level, then we do that & take the core

  → this ensures that δ_i are Woodin in M

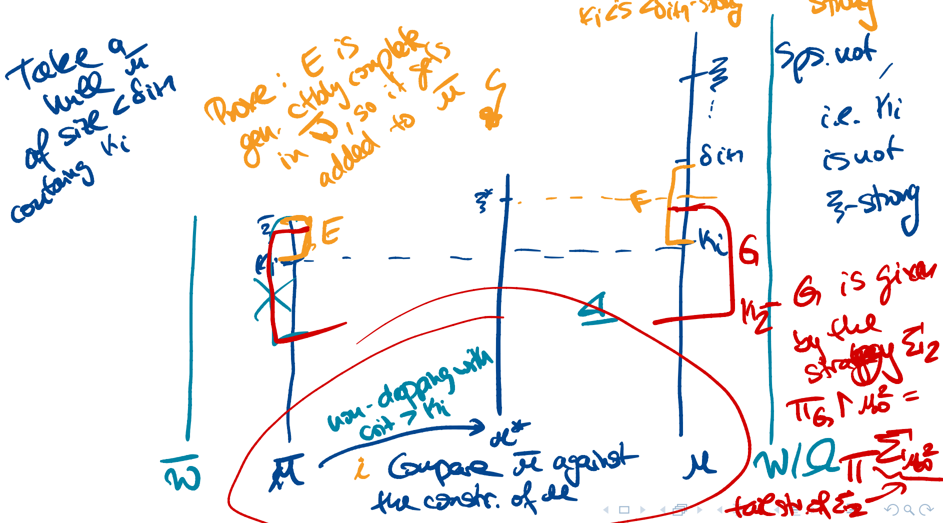
- if possible, add extenders with crit κ_i, that are generically ctbly complete

  → Aim: these will witness strongness

gen. ctbly complete b/c we look for an iterable model

Part II: The translation procedure

Why does the construction work?



Summary

We have seen a deep connection between determinacy, inner models with large cardinals, and hybrid mice.

Corollary (Larson-Sargsyan-Wilson, M)

The following theories are equiconsistent.

- T_1 $\mathrm{ZF} + \mathrm{AD}_{\mathbb{R}} +$ *all sets of reals are universally Baire.*
- T_2 $\mathrm{ZFC} +$ *there is a limit of Woodin cardinals that is a limit of strong cardinals.*

Question: Take T_2: ZFC + there is a Woodin limit of Woodins. What should T_1 be?

Links to the images:

- https://pixabay.com/illustrations/chess-play-relax-think-chess-board-1019908/
- https://pixabay.com/illustrations/friends-kameraden-camaraderie-good-1013882/
- https://pixabay.com/vectors/helical-helix-mouse-spiral-1294937/
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