Integration with filters

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Introduction

Feynman introduced a formalism in quantum mechanics in which one averages over all possible random paths a particle can take. Giving this a proper mathematical treatment in terms of classical measure theory has been challenging. The basic issue is described by Charlie Wood in a recent *Quanta* article:

> *"No known mathematical procedure can meaningfully average an infinite number of objects covering an infinite expanse of space in general. The path integral is more of a physics philosophy than an exact mathematical recipe."*

We present here such a general procedure.

One approach is with nonstandard analysis. Suppose X is a set and G is a divisible Abelian group. Suppose $j : V \to M$ is an elementary embedding such that in M, there is Y such that $M \models$ "Y is finite", and $Y \supseteq j[X]$. For $f : X \to G$ we can average f by computing in M:

$$|Y|^{-1} \sum_{i \in Y} j(f)(i) \in j(G)$$

For $G = \mathbb{R}$, we can extract the *standard part* of $x \in j(\mathbb{R})$:

$$\text{st}(x) = \sup\{q \in \mathbb{Q} : q < x\}$$

Theorem (Henson, 1972)

Suppose (X, μ) is a probability space. There is some nonstandard extension $j : V \to M$ and an M-finite $Y \supseteq j[X]$ such that for all integrable $f : X \to \mathbb{R}$,

$$\int f \, d\mu = \text{st}\left(|Y|^{-1} \sum_{i \in Y} j(f)(i)\right)$$

Typically such an embedding is obtained using an ultrafilter. We aim towards a more constructive approach using reduced powers via fine filters.

A feature of this kind of approach is that it allows for a more fine-grained quantification of the sizes of sets and the behavior of functions. For example, we can have a series of relations among sets A_i, B_i like:

$$m(A_i) \ll m(A_{i+1});\ m(B_i) \approx r_i m(A_i),$$

for $i \in \mathbb{N}$ and positive reals r_i. The empirical meaning can be cashed out by saying that for all $i, n \in \mathbb{N}$, a generic finite sample of points z will have

$$\frac{|z \cap A_i|}{|z \cap A_{i+1}|} + \left| \frac{|z \cap B_i|}{|z \cap A_i|} - r_i \right| < \frac{1}{n}$$

Classical measures would flatten the description to just give all of these sets measure zero.

Comparison rings

Let us say that a structure is a *comparison ring* if it is commutative ring with 1 and it carries a binary relation $<$ with the following properties:

1. $<$ is a strict partial order (i.e. transitive and irreflexive).
2. For all a, b, c, if $a < b$, then $a + c < b + c$.
3. For all a, b, if $a, b > 0$, then $ab > 0$.
4. For all a, a has a multiplicative inverse if and only if $a^2 > 0$.

Proposition

Suppose K is a comparison ring and $a, b, c, d \in K$.

1. $K \models 0 < 1$.
2. If $a > 0$, then a is invertible and $a^{-1} > 0$.
3. If $a < 0$, then a is invertible and $a^{-1} = -(-a)^{-1} < 0$.
4. If $a < b$ and $c < d$, then $a + c < b + d$.
5. If $a < b$ and $0 < c$, then $ac < bc$.
6. $0 < a < b$ if and only if $0 < b^{-1} < a^{-1}$.
7. The ordered field $\mathbb{Q}$ of rational numbers is a substructure of K.

Some terminology

Let K be a comparison ring, and let $a, b \in K$.

- We say a is *finite* when $-n < a < n$ for some $n \in \mathbb{N}$, and *infinite* when it is not finite. Note that the set of finite elements forms a subring.
- If $b > 0$ and $-b < na < b$ for all $n \in \mathbb{Z}$, then we write $a \ll b$. Note that the set $\{a \in K : a \ll b\}$ is closed under addition and under multiplication by finite elements.
- We say a is *infinitesimal* when $a \ll 1$.
- We say $a \sim b$ when $a - b$ is infinitesimal.
- We say $a \approx b$ when b is invertible and $ab^{-1} \sim 1$. Note that this implies a is also invertible, because $1/4 < (ab^{-1})^2$ and so $0 < b^2/4 < a^2$. Thus also $ba^{-1} \sim 1$.
- We say that $a, b > 0$ are *Archimedean-equivalent* if there are $n, m \in \mathbb{N}$ such that $a < nb$ and $b < ma$.

For a structure $\mathfrak{A}$, a set Z, and a filter F over Z, we write $\text{Pow}(\mathfrak{A}, F)$ for the reduced power of all functions $f : Z \to \mathfrak{A}$, where we say f, g are equivalent modulo F if they are equal on a set $S \in F$. We interpret the language of $\mathfrak{A}$ modulo F similarly.

If K is an ordered field, then usually $\text{Pow}(K, F)$ is not an ordered field, because if F is not maximal, we lose the existence of multiplicative inverses for all nonzero elements and the totality of the ordering. However:

Lemma

If K is a comparison ring and F is a filter over a set Z, then $\text{Pow}(K, F)$ is also a comparison ring.

For a comparison ring K, we define the upper standard part of $a \in K$ as $\text{st}^+ a = \inf\{q \in \mathbb{Q} : a < q\}$ and the lower standard part of a as $\text{st}^- a = \sup\{q \in \mathbb{Q} : a > q\}$.

We say that $a \in K$ has a standard part if the upper standard part and the lower standard part are equal. In this case, we define $\text{st}\, a = \text{st}^+ a = \text{st}^- a$.

Filter integrals

Let G be a divisible Abelian group and F a fine filter over $[X]^{<\omega}$. We define an operator that assigns to functions $f : X \to G$ a value in $\mathrm{Pow}(G, F)$.

$$\int f\,dF := \left[z \mapsto \sum_{x \in z} f(x)/|z|\right]_F .$$

We have that for any $c \in G$, $\int c\,dF = [c]_F$. Furthermore, for any functions $f, g : X \to G$, $\int (f+g)\,dF = \int f\,dF + \int g\,dF$. Moreover, when G has a ring structure, the integral is a linear operator.

If K is a comparison ring and $f : X \to K$ is such that $\int f\,dF$ has a standard part, then we write this "standard integral" as $⨎ f\,dF$.

The set $\{f \in \mathrm{Fun}(X, K) : f \text{ has a standard integral}\}$ is a vector space over $\mathbb{Q}$. If $K \supseteq \mathbb{R}$, then $\mathbb{Q}$ can be replaced with $\mathbb{R}$.

Representations

Lemma

Suppose μ is a finitely additive measure defined on an algebra $\mathcal{A}$ of subsets of an infinite set X, taking extended real values in $[0, \infty]$ and giving measure zero to all singletons. Let $Y_1, \ldots, Y_k \in \mathcal{A}$ have finite measure, let $x_1, \ldots, x_l \in X$, and let $n \in \mathbb{N}$ be positive. There exists a finite $z \subseteq X$ that satisfies the following properties:

1. $x_1, \ldots, x_l \in z$;
2. $nl < |z|$;
3. *if* $\mu(Y_1 \cup \cdots \cup Y_k) > 0$, *then* $z \setminus \{x_1, \ldots, x_l\} \subseteq Y_1 \cup \cdots \cup Y_k$;
4. *for* $1 \leq i, j \leq k$, *if* $\mu(Y_i) \neq 0$, *then:*

$$\left| \frac{|z \cap Y_j|}{|z \cap Y_i|} - \frac{\mu(Y_j)}{\mu(Y_i)} \right| < \frac{1}{n}$$

Theorem

Suppose μ is a finitely additive real-valued atomless probability measure defined on an algebra $\mathcal{A}$ of subsets of X. Then there is a definable filter F_μ over $[X]^{<\omega}$, which is the smallest fine filter F with the property that for any bounded μ-measurable function $f : X \to \mathbb{R}$,

$$\int f\, d\mu = \rlap{\square}\int f\, dF.$$

If μ is countably additive, then the same conclusion holds for all integrable functions f.

Proposition

Suppose μ is a countably additive complete probability measure defined on a σ-algebra $\mathcal{A} \subseteq \mathcal{P}(X)$. Let $f : X \to \mathbb{R}$ be bounded. The following are equivalent:

1. f is μ-measurable.
2. f has a standard F_μ-integral.

Iterated integrals

Suppose we have fine filters F, G over $[X]^{<\omega}, [Y]^{<\omega}$ respectively. We construct a fine filter $F \times G$ over $[X \times Y]^{<\omega}$ concentrating on the finite rectangles, $F \times G$ is the set of $A \subseteq [X \times Y]^{<\omega}$ such that

$$\{z_1 \in [Y]^{<\omega} : \{z_0 \in [X]^{<\omega} : z_0 \times z_1 \in A\} \in F\} \in G.$$

Let $\mathfrak{A}$ be any algebraic structure. Then there is a canonical isomorphism

$$\iota : \mathrm{Pow}(\mathfrak{A}, F \times G) \cong \mathrm{Pow}(\mathrm{Pow}(\mathfrak{A}, F), G).$$

Lemma

Suppose K is a divisible Abelian group, F, G are fine filters over $[X]^{<\omega}, [Y]^{<\omega}$ respectively. Then for all $f : X \times Y \to K$,

$$\int f \, d(F \times G) = \iint f \, dF dG.$$

Proposition

Suppose F, G are fine filters over $[X]^{<\omega}, [Y]^{<\omega}$ respectively. For $A \subseteq X$ and $B \subseteq Y$,

$$\iint^{+} \chi_{A \times B}\, dF dG = \left(\int^{+} \chi_A\, dF \right) \left(\int^{+} \chi_B\, dG \right);$$

$$\iint^{-} \chi_{A \times B}\, dF dG = \left(\int^{-} \chi_A\, dF \right) \left(\int^{-} \chi_B\, dG \right).$$

Theorem

Suppose μ, ν are countably additive probability measures on X, Y respectively. Then for all $\mu \times \nu$-integrable functions $f : X \times Y \to \mathbb{R}$, there are sets $A \subseteq X$ and $B \subseteq Y$ such that $\mu(A) = \nu(B) = 1$, and

$$\int f\, d(\mu \times \nu) = \int_{A \times B} f\, d(F_\mu \times F_\nu) = \int_{B \times A} \bar{f}\, d(F_\nu \times F_\mu).$$

Non-Archimedean measures and geometry

A well-known no-go result in functional analysis states that there is no analogue of Lebesgue measure on infinite-dimensional separable Banach spaces such that:

- every Borel set is measurable;
- the measure is translation-invariant;
- every point has a neighborhood with finite measure.

This result is based on the following more general fact: If X is an infinite-dimensional normed vector space over the reals, then every open ball contains an infinite collection of pairwise-disjoint open balls of equal radius (in fact only $1/4$ the radius of the original ball). Thus there cannot exist even a finitely additive translation-invariant measure on an infinite-dimensional normed real vector space that gives every open ball of finite radius a positive real measure.

We a construct a non-Archimedean measure on a concrete space that contrasts with this impossibility result.

Let us consider the space $\mathbb{R}^{<\omega}$ of ω-sequences of real numbers that are eventually zero. Each $\mathbb{R}^n$ appears canonically as the collection of sequences $\vec{x}$ such that $\vec{x}(m) = 0$ for all $m \geq n$. Of course, this real vector space comes along with the standard Euclidean norm.

For a positive integer n, let μ_n be the Lebesgue measure on $\mathbb{R}^n$. Let us call a set $A \subseteq \mathbb{R}^{<\omega}$ *middling* if for all but finitely many $n < \omega$, $\mu_n(A \cap \mathbb{R}^n) < \infty$, and for infinitely many $n < \omega$, $\mu_n(A \cap \mathbb{R}^n) > 0$. Intuitively, middling sets are larger than finite dimensional sets but much smaller than the whole space. Clearly, every open ball in $\mathbb{R}^{<\omega}$ is middling.

Theorem

There is a fine filter Γ *over* $[\mathbb{R}^{<\omega}]^{<\omega}$ *and a* $\ll$*-increasing sequence of positive infinitesimals* $\langle \varepsilon_i : i < \omega \rangle \subseteq \text{Pow}(\mathbb{R}, \Gamma)$*, such that, if* $m(A) = \int \chi_A d\Gamma$ *for* $A \subseteq \mathbb{R}^{<\omega}$*, then:*

1. $\varepsilon_n = m([0,1]^n)$*, the measure of the n-dimensional unit cube.*
2. *For any positive-volume Borel subset A of an n-dimensional piecewise smooth surface S,* $m(A) \approx \text{vol}_n(A)\varepsilon_n$*.*
3. *For any countable* $C \subseteq \mathbb{R}^{<\omega}$*,* $m(C) \ll \varepsilon_1$*.*
4. *For any middling Borel* $A \subseteq \mathbb{R}^{<\omega}$ *and any* $\vec{x} \in \mathbb{R}^{<\omega}$*,* $m(A + \vec{x}) \approx m(A)$*.*

If κ is a cardinal such that every set of reals of size $< \kappa$ has Lebesgue measure zero, then we can replace "countable" with "$<\kappa$-sized" in the third item. Let us call the resulting filter Γ_κ.

Conditional probabilities

Given a fine filter F on $[X]^{<\omega}$, we define:

- the expected value of a function f as $\mathrm{E}(f) := \int f \, dF$.
- the conditional expectation of a function f on a nonempty set $A \subseteq X$ as $\mathrm{E}(f|A) := \mathrm{E}(f\chi_A)/\,\mathrm{E}(\chi_A)$. Note that this is well-defined since $\int \chi_A \, dF > 0$.
- the probability of a set A as $\Pr(A) = \mathrm{E}(\chi_A)$ and the conditional probability of A given B as $\Pr(A|B) = \mathrm{E}(\chi_A|B)$.

Our filter Γ on $\mathbb{R}^{<\omega}$ gives very simple answers to questions like: What is the probability that you are in the arctic circle given that you are on the prime meridian? (Answer: about 13%.) Historically there was some debate about conditioning on events with classical measure zero, since geometric paradoxes can arise if we try to derive conditional probability on lines from a background probability distribution on the plane. Kolmogorov asserted that the above question about the sphere doesn't make sense, but it is unproblematic with filter-probabilities.

Dimension

As it accords with classical geometry, it seems natural to define the dimension of an arbitrary subset A of $\mathbb{R}^{<\omega}$ to be the Archimedean equivalence class of $\int \chi_A \, d\Gamma$.

Suppose F is a fine filter over $[X]^{<\omega}$. Let $\dim_F(A)$ denote the Archimedean class of $\int \chi_A \, dF$. Let us say $\dim_F(A) < \dim_F(B)$ when $\int \chi_A \, dF \ll \int \chi_B \, dF$. Note that if $F' \supseteq F$, then $\dim_F(A) < \dim_F(B)$ implies $\dim_{F'}(A) < \dim_{F'}(B)$.

Let us say that a set $A \subseteq X$ is *F-solid* if for all $Y \subseteq X$ such that $|Y| < |X|$, $\dim_F(Y) < \dim_F(A)$. If Martin's Axiom holds, then each positive-volume Borel subset of a finite dimensional surface in $\mathbb{R}^{<\omega}$ is $\Gamma_{\mathfrak{c}}$-solid.

Lemma

Assume MA. Let F be a fine filter over $[\mathfrak{c}]^{<\omega}$ that is generated by a base of size $\mathfrak{c}$. Suppose $\{A_\alpha : \alpha < \mathfrak{c}\}$ and $\{B_\alpha : \alpha < \mathfrak{c}\}$ are collections of subsets of $\mathfrak{c}$ such that each B_α is F-solid, and for all $\alpha, \beta < \mathfrak{c}$, $\dim_F(A_\alpha) < \dim_F(B_\beta)$. Then there is a filter $F' \supseteq F$ with a base of size $\mathfrak{c}$ and an F'-solid $C \subseteq \mathfrak{c}$ such that for all $\alpha, \beta < \mathfrak{c}$, $\dim_{F'}(A_\alpha) < \dim_{F'}(C) < \dim_{F'}(B_\beta)$.

Proof:

Let $\langle X_\alpha : \alpha < \mathfrak{c}\rangle$ be an enumeration of a base for F. Let $\langle M_\alpha : \alpha < \mathfrak{c}\rangle$ be a sequence of elementary submodels of $H_{(2^{\mathfrak{c}})^+}$ such that:

- For each $\alpha < \mathfrak{c}$, $|M_\alpha| < \mathfrak{c}$, $M_\alpha \cap \mathfrak{c}$ is an ordinal, and $M_\alpha \in M_{\alpha+1}$.
- For each limit $\lambda < \mathfrak{c}$, $M_\lambda = \bigcup_{\alpha<\lambda} M_\alpha$.
- $F, \{(A_\alpha, B_\alpha, X_\alpha) : \alpha < \mathfrak{c}\} \in M_0$.

Claim

Suppose $\delta < \mathfrak{c}$, $s \in [\mathfrak{c}]^{<\omega}$, and $n \geq 2$. For $p \in \mathrm{Add}(\omega, \mathfrak{c})$, let $C_p = \{\beta \in \mathrm{dom}(p) : p(\beta) = 1\}$. Consider the set

$$D_{\delta,s,n} = \{p : \mathrm{dom}(p) \in \bigcap_{i \in s} X_i, \text{ and for all } i, j \in s$$

$$n\,(|\,\mathrm{dom}(p) \cap A_i| + |\,\mathrm{dom}(p) \cap \delta|) < |C_p \setminus \delta| < n^{-1}|\,\mathrm{dom}(p) \cap B_j|\}.$$

Then $D_{\delta,s,n}$ is dense.

Hint: Find $z \in \bigcap_{i \in s} X_i$ such that $z \supseteq \mathrm{dom}(p)$, $|z| > 2|\,\mathrm{dom}(p)|$, and for all $\alpha, \beta \in s$,

$$2n^2(|s||z \cap A_\alpha| + |z \cap \delta|) < |z \cap B_\beta|.$$

By MA, let G_0 be $\mathrm{Add}(\omega, \mathfrak{c})$-generic over M_0. Let $C_0 = \{\gamma : G_0(\gamma) = 1\}$. Assume inductively that we have a sequence of sets $\langle C_\alpha \subseteq M_\alpha : \alpha < \beta \rangle$, with $C_\alpha \cap M_{\alpha'} = C_{\alpha'}$ for $\alpha' < \alpha$. If β is a limit, let $C_\beta = \bigcup_{\alpha<\beta} C_\alpha$. If $\beta = \beta' + 1$, let G_β be $\mathbb{P}$-generic over M_β, and let

$$C_\beta = C_{\beta'} \cup \{\gamma : \gamma > M_{\beta'} \cap \mathfrak{c}, G_\beta(\gamma) = 1\}.$$

Finally, we let $C = \bigcup_{\alpha<\mathfrak{c}} C_\alpha$.

We can show that for each $\delta < \mathfrak{c}$, each $s \in [\mathfrak{c}]^{<\omega}$, and each positive $n \in \mathbb{N}$, there is $z \in \bigcap_{i\in s} X_i$ such that for $\alpha, \beta \in s$,

$$n(|z \cap A_\alpha| + |z \cap \delta|) < |z \cap C| < n^{-1}|z \cap B_\beta|.$$

This means that the following family has the finite intersection property:

- $\{z : n|z \cap A_\beta| < |z \cap C|\}$ for $n < \omega$ and $\alpha < \mathfrak{c}$;
- $\{z : n|z \cap C| < |z \cap B_\beta|\}$ for $n < \omega$ and $\beta < \mathfrak{c}$;
- $\{z : n|z \cap \gamma| < |z \cap C|\}$ for $n < \omega$ and $\gamma < \mathfrak{c}$;
- X_δ for $\delta < \mathfrak{c}$.

Let F' be the generated filter. Then C is F'-solid, and for $\alpha, \beta < \mathfrak{c}$, $\dim_{F'}(A_\alpha) < \dim_{F'}(C) < \dim_{F'}(B_\beta)$.

Applying the lemma inductively, we get:

Theorem

Assume MA and $2^{\mathfrak{c}} = \mathfrak{c}^+$. There is an extension of $\Gamma_{\mathfrak{c}}$ to an ultrafilter U such that for any collections $\mathcal{S}, \mathcal{T} \subseteq \mathbb{R}^{<\omega}$ of size at most $\mathfrak{c}$ such that $\dim_U(S) < \dim_U(T)$ for $S \in \mathcal{S}$ and $T \in \mathcal{T}$ and each $T \in \mathcal{T}$ is U-solid, there is a U-solid C such that $\dim_U(S) < \dim_U(C) < \dim_U(T)$ for all $S \in \mathcal{S}$ and $T \in \mathcal{T}$.

Consequently, for any sets A, B such that B is solid and $\dim_U(A) < \dim_U(B)$, the collection of dimensions of U-solid sets in the open interval $(\dim_U(A), \dim_U(B))$ does not have a coinitial or cofinal set of size $\mathfrak{c}$.

Thanks for your attention!