Distributivity spectrum of forcing notions

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Definition

$\mathbb{P}$ is λ-distributive if it does not add a function $f : \lambda \to Ord$ with $f \notin V$.
$\mathfrak{h}(\mathbb{P}) :=$ least λ such that $\mathbb{P}$ is not λ-distributive (the distributivity of $\mathbb{P}$).

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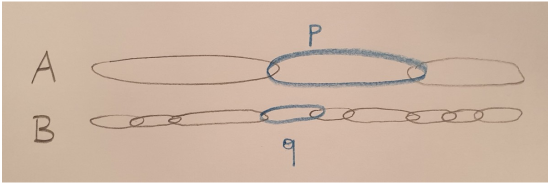
For maximal antichains A and B,

$$B \text{ refines } A :\Longleftrightarrow \forall q \in B \, \exists p \in A \, (q \leq p).$$



Proposition

$\mathbb{P}$ is λ-distributive if and only if for each family $\mathcal{A} = \{A_\xi : \xi < \lambda\}$ of maximal antichains in $\mathbb{P}$, there exists a common refinement (i.e., a maximal antichain B such that B refines A_ξ for each $\xi < \lambda$).

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$\mathfrak{h}(\mathbb{P}) :=$ least λ such that $\mathbb{P}$ is not λ-distributive (the distributivity of $\mathbb{P}$).

Let $\lambda = \mathfrak{h}(\mathbb{P})$, and let $f : \lambda \to \mathit{Ord}$ witness this, i.e., $f \notin V$.
Note that $f \restriction \gamma \in V$ for every $\gamma < \lambda$ (f is not just new, but even "fresh").

Definition (Fresh function spectrum)

We say that $\lambda \in \text{FRESH}(\mathbb{P})$ if in some extension of V by $\mathbb{P}$,

there exists a fresh function on λ,

i.e., a function $f : \lambda \to \mathit{Ord}$ with

1. $f \notin V$, but
2. $f \restriction \gamma \in V$ for every $\gamma < \lambda$.

Note: $\lambda \in \text{FRESH}(\mathbb{P}) \iff \mathrm{cf}(\lambda) \in \text{FRESH}(\mathbb{P})$

So from now on, we only talk about regular cardinals λ.

- $\min(\mathrm{FRESH}(\mathbb{P})) = \mathfrak{h}(\mathbb{P})$

Proposition

If $\lambda > |\mathbb{P}|$, then $\lambda \notin \mathrm{FRESH}(\mathbb{P})$.

Proof (Sketch).

- assume towards contradiction that $\dot{f} : \lambda \to \mathit{Ord}$ is fresh
- for each $\gamma < \lambda$, fix $p_\gamma \in \mathbb{P}$ such that p_γ decides $\dot{f} \restriction \gamma$
- λ regular, so there exists p^* with $p_\gamma = p^*$ for unboundedly many γ
- so p^* decides $\dot{f}$ (and hence $\dot{f}$ is not new) □

- $\mathrm{FRESH}(\mathbb{P}) \subseteq [\mathfrak{h}(\mathbb{P}), |\mathbb{P}|]$

Example: Let $\mathbb{C}$ be the usual ω-Cohen forcing (with $|\mathbb{C}| = \omega$).

$$\mathrm{FRESH}(\mathbb{C}) = \{\omega\}$$

Let $\mathbb{C}_\mu$ be the forcing for adding μ many ω-Cohen reals (μ arbitrary).

$$\mathsf{FRESH}(\mathbb{C}_\mu) = ?$$

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$$\mathsf{FRESH}(\mathbb{C}_\mu) = \{\omega\}$$

Proposition

If $\mathbb{P}$ satisfies $\mathbb{P} \times \mathbb{P}$ is δ-c.c. and $\lambda \geq \delta$, then $\lambda \notin \mathsf{FRESH}(\mathbb{P})$.

Is $\mathbb{P}$ being δ-c.c. sufficient? No: consider a Suslin tree T (on ω_1)

- T is c.c.c. (i.e., ω_1-c.c.)
- but $\omega_1 \in \mathsf{FRESH}(T)$
  - the generic branch is a fresh function on ω_1
- $\mathsf{FRESH}(T) = \{\omega_1\}$, since...
  - T is ω-distributive
  - $|T| = \omega_1$

Proposition

If $\mathbb{P}$ is δ-c.c. and $\lambda > \delta$, then $\lambda \notin \mathsf{FRESH}(\mathbb{P})$.

Proposition

If $\mathbb{P}$ collapses λ to $\mathfrak{h}(\mathbb{P})$, then $\lambda \in \text{FRESH}(\mathbb{P})$.

It follows that $[\mathfrak{h}(\mathbb{P}), \lambda] \subseteq \mathit{FRESH}(\mathbb{P})$.

Example (under CH): Let $\mathit{Coll}(\omega_1, \omega_2)$ be the forcing collapsing ω_2 to ω_1 with countable conditions.

$$\text{FRESH}(\mathit{Coll}(\omega_1, \omega_2)) = \{\omega_1, \omega_2\}$$

Theorem (Base Matrix Theorem; Balcar-Pelant-Simon)

$\mathcal{P}(\omega)/\text{fin}$ collapses $\mathfrak{c}$ to $\mathfrak{h}$.

$$\mathfrak{h} := \mathfrak{h}(\mathcal{P}(\omega)/\mathit{fin})$$

Corollary

$\text{FRESH}(\mathcal{P}(\omega)/\text{fin}) = [\mathfrak{h}, \mathfrak{c}]$.

Possible fresh function spectra under GCH

Let A be a set of regular cardinals.

Does there exist a forcing $\mathbb{P}$ such that FRESH$(\mathbb{P}) = A$?

Examples under GCH:

- FRESH$(\mathbb{C}(\lambda)) = \{\lambda\}$
  - $<\lambda$-closed
  - $|\mathbb{C}(\lambda)| = \lambda$
- FRESH$(\mathbb{C} \times \mathbb{C}(\omega_1)) = \{\omega, \omega_1\}$
  - If a function is fresh, it remains fresh in any extension.
- FRESH$(\mathbb{C}(\omega_1) \times \mathbb{C}(\omega_3)) = \{\omega_1, \omega_3\}$
  - $\mathbb{C}(\omega_1) \times \mathbb{C}(\omega_3) \cong \mathbb{C}(\omega_3) * \check{\mathbb{C}}(\omega_1) \cong \mathbb{C}(\omega_3) * \mathbb{C}(\omega_1)$
- If A is finite: FRESH$(\prod_{\lambda \in A} \mathbb{C}(\lambda)) = A$

What if A is infinite?

Definition

A set A of regular cardinals is an Easton set if for every limit point α of A,

- α regular $\Rightarrow \alpha \in A$,
- α singular $\Rightarrow \alpha^+ \in A$.

The Easton closure of a set is the smallest superset which is an Easton set.

Let ${}^E\prod_{\lambda \in A} \mathbb{P}_\lambda$ denote the Easton product of the $\mathbb{P}_\lambda$:

- full support at singular limits,
- bounded support at regular limits (i.e., inaccessibles).

Theorem (GCH)

Let A be a set of regular cardinals. Then $\mathrm{FRESH}({}^E\prod_{\lambda \in A} \mathbb{C}(\lambda))$ is equal to the Easton closure of A.

If A is an Easton set, then there exists $\mathbb{P}$ with $\mathit{FRESH}(\mathbb{P}) = A$.

Theorem (GCH)

Let A be a set of regular cardinals. Then $\mathrm{FRESH}({}^{E}\prod_{\lambda\in A}\mathbb{C}(\lambda))$ is equal to the Easton closure of A.

Regular limit: bounded support product in regular limit α adds α-Cohen

Singular limit:

Theorem (GCH)

Let α be singular and $A \subseteq \alpha$ be an unbounded subset of regular cardinals. Then ${}^{E}\prod_{\lambda\in A}\mathbb{C}(\lambda)$ adds an α^+-Cohen real.

In particular, $\alpha^+ \in \mathit{FRESH}({}^{E}\prod_{\lambda\in A}\mathbb{C}(\lambda))$.

Proof.

Follows from a paper of Shelah, which uses pcf theory. □

Are unbounded subsets $A \subseteq \aleph_\omega$ realizable?

Since $\aleph_{\omega+1} \notin A$, they are not Easton sets!

For $A = \{\aleph_n : n \in \omega\}$, yes:

- either $Coll(\aleph_0, \aleph_\omega)$ or finite support product of Cohen forcings work.

Question (GCH)

Let $A \subsetneq \aleph_\omega$ unbounded. Is there a forcing $\mathbb{P}$ with $\text{FRESH}(\mathbb{P}) = A$?

$\sim$

Let α be regular limit, i.e, inaccessible.

Question (GCH)

Let $A \subseteq \alpha$ unbounded. Is there a forcing $\mathbb{P}$ with $\text{FRESH}(\mathbb{P}) = A$?

Not even clear if A is the set of all regular cardinals strictly below α.

Question (GCH)

What does the full support product of the $\mathbb{C}(\lambda)$ for $\lambda < \alpha$ do?

Omitting fresh function spectra

Let A be a set of regular cardinals.

Does there (always?) exist a forcing $\mathbb{P}$ *such that FRESH*$(\mathbb{P}) = A$*?*

Remember:

$$\text{FRESH}(\mathbb{C}) = \{\omega\}$$

Is there a forcing $\mathbb{P}$ with FRESH$(\mathbb{P}) = \{\omega_1\}$?

Yes, if:

- CH holds (then $\mathbb{C}(\omega_1)$ does the job),
- there exists a Suslin tree (then the Suslin tree does the job).

Is it provable in ZFC?

Definition

Todorčević's maximality principle is the following assertion:

If $\mathbb{P}$ is a forcing which adds a fresh subset of ω_1, then

- $\mathbb{P}$ collapses ω_1, or
- $\mathbb{P}$ collapses ω_2.

It is consistent, relative to the existence of an inaccessible cardinal.

Theorem

Assume Todorčević's maximality principle, and "$0^\#$ does not exist".

Then for every forcing with $\omega_1 \in \text{FRESH}(\mathbb{P})$,

- $\omega \in \text{FRESH}(\mathbb{P})$, or
- $\omega_2 \in \text{FRESH}(\mathbb{P})$.

In particular, there is no forcing $\mathbb{P}$ with FRESH$(\mathbb{P}) = \{\omega_1\}$.

Let $\mathbb{M}$ denote Mathias forcing, and let $\mathbb{S}$ denote Sacks forcing.

Proposition

$\mathrm{FRESH}(\mathbb{M}) = \{\omega\} \cup [\mathfrak{h}, \mathfrak{c}]$.

Proof is based on $\mathbb{M} \cong \mathcal{P}(\omega)/\mathrm{fin} * \mathbb{M}(G)$.

Proposition

$\mathrm{FRESH}(\mathbb{S}) = \{\omega\}$.

Question

Is there a forcing $\mathbb{P}$ which is minimal, yet $|\mathrm{FRESH}(\mathbb{P})| \geq 2$?

Question

Can Laver forcing (or Miller forcing) be represented as two-step iteration?

Recall that (under CH)

- $\mathsf{FRESH}(\mathbb{C} \times \mathbb{C}(\omega_1)) = \{\omega, \omega_1\}$

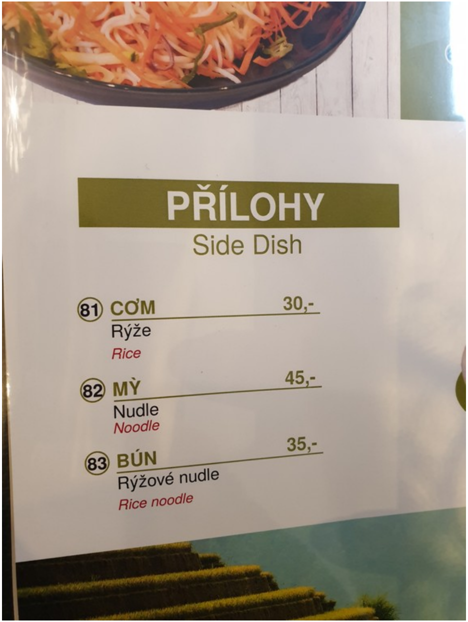
However, the following holds:

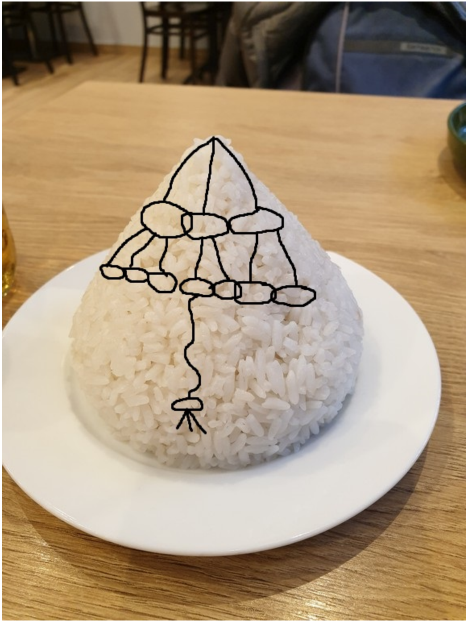
- $\mathsf{FRESH}(\mathbb{C} * \mathbb{C}(\omega_1)) = \{\omega\}$
  - the generic ω_1-Cohen added by the second iterand is not fresh over V

Proposition

Let $\dot{\mathbb{Q}}$ be a $\mathbb{C}$-name for a σ-strategically closed forcing. Then

$$\mathsf{FRESH}(\mathbb{C} * \dot{\mathbb{Q}}) = \{\omega\}.$$


PŘÍLOHY
Side Dish
81 CƠM 30,-
Rýže
Rice
82 MỲ 45,-
Nudle
Noodle
83 BÚN 35,-
Rýžové nudle
Rice noodle



Definition (Combinatorial distributivity spectrum)

We say $\mathcal{A} = \{A_\xi : \xi < \lambda\}$ is a distributivity matrix of height λ (for $\mathbb{P}$) if

- A_ξ is a maximal antichain in $\mathbb{P}$ (for each $\xi < \lambda$),
- A_η refines A_ξ whenever $\eta \geq \xi$,
  - A_η refines A_ξ $:\Longleftrightarrow \forall q \in A_\eta \, \exists p \in A_\xi \, (q \leq p)$
- there is no common refinement
  - i.e., there is no maximal antichain B which refines every A_ξ.

We say $\lambda \in \mathrm{COM}(\mathbb{P})$ if there is a distributivity matrix of height λ for $\mathbb{P}$.

Is $\mathrm{COM}(\mathbb{P}) = \mathrm{FRESH}(\mathbb{P})$?

Proposition

$\min(\mathrm{COM}(\mathbb{P})) = \min(\mathrm{FRESH}(\mathbb{P})) = \mathfrak{h}(\mathbb{P})$

Proposition

$\mathrm{COM}(\mathbb{P}) \subseteq \mathrm{FRESH}(\mathbb{P})$

Proposition

$\mathrm{COM}(\mathbb{P}) = \mathrm{FRESH}(\mathbb{P})$ in case $\mathbb{P}$ is a complete Boolean Algebra

But note:

The Boolean algebra $\mathcal{P}(\omega)/\mathrm{fin}$ is not complete!!

$$\{\mathfrak{h}\} \subseteq \mathrm{COM}(\mathcal{P}(\omega)/\mathrm{fin}) \subseteq \mathrm{FRESH}(\mathcal{P}(\omega)/\mathrm{fin}) = [\mathfrak{h}, \mathfrak{c}]$$

Theorem

Let V_0 be a model of ZFC which satisfies GCH. In V_0, let

$$\omega_1 < \lambda \leq \mu$$

be cardinals such that λ is regular and $\mathrm{cf}(\mu) > \omega$. Then there is a c.c.c. (and hence cofinality preserving) extension W of V_0 such that

$$W \models \omega_1 = \mathfrak{h} = \mathfrak{b} \ \wedge \ \lambda \in \mathsf{COM}(\mathcal{P}(\omega)/\mathsf{fin}) \ \wedge \ \mu = \mathfrak{c}.$$

1. go to the usual Cohen model with $\mu = \mathfrak{c}$
2. forcing (iteration) to add a distributivity matrix of height λ
3. show that $\omega_1 = \mathfrak{h} = \mathfrak{b}$ in the final model

Corollary

It is consistent that $\mathfrak{h} < \mathfrak{c} = \omega_2$, and

$$\{\omega_1, \omega_2\} = \mathsf{COM}(\mathcal{P}(\omega)/\mathsf{fin}) = \mathsf{FRESH}(\mathcal{P}(\omega)/\mathsf{fin}) = [\mathfrak{h}, \mathfrak{c}]$$

Our definition is inspired by Hechler's forcings defined in *"Short complete nested sequences in* $\beta\mathbb{N}\setminus\mathbb{N}$ *and small maximal almost-disjoint families"*.

Definition

Let $T := \{\sigma \in \lambda^{<\lambda} : \sigma$ has successor length$\}$.

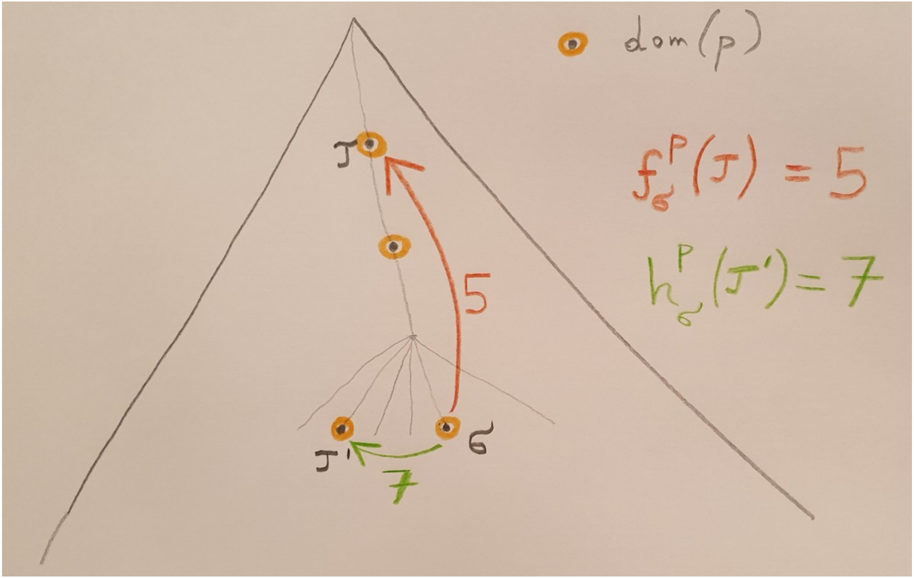
Define a forcing $\mathbb{Q}$ as follows: p is a condition if

- p is a finite function with $\mathrm{dom}(p) \subseteq T$,
- for each $\sigma \in \mathrm{dom}(p)$, $p(\sigma) = (s^p_\sigma, f^p_\sigma, h^p_\sigma)$, with $s^p_\sigma \in 2^{<\omega}$.

If G is a generic filter, let

- $a_\sigma := \bigcup_{p \in G} s^p_\sigma$ (which is an infinite subset of ω),
- $A_{\xi+1} := \{a_\sigma : \sigma$ has length $\xi + 1\}$ (for $\xi < \lambda$),
- $\mathcal{A} := \{A_{\xi+1} : \xi < \lambda\}$ is the intended generic distributivity matrix.

f^p_σ: promises for the $\subseteq^*$-relation
h^p_σ: promises for the almost disjointness


dom(p)
$f_\sigma^p(\tau) = 5$
$h_\sigma^p(\tau') = 7$
τ
5
τ'
σ
7

Problem

The levels of the generic matrix are not mad families (new reals are added).

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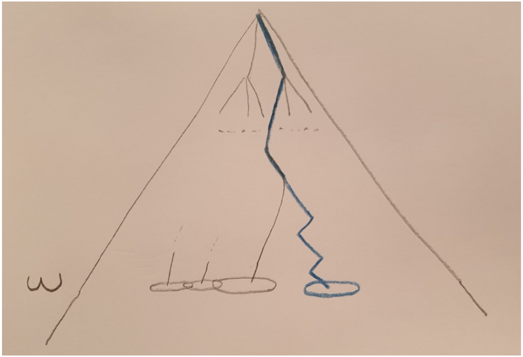

Solution

Use iterated forcing to make sure that the levels are maximal in the end.

- finite support iteration of length λ

A distributivity matrix of height λ is added

Finite support iteration of length λ yields distributivity matrix of height λ.

Main steps of the proof:

- iterands have ccc
- each node of $\lambda^{<\lambda}$ appears at some intermediate stage of the iteration
  - so in the final model, a_σ is defined for all $\sigma \in \lambda^{<\lambda}$
- along branches through $\lambda^{<\lambda}$, we have $\subseteq^*$-decreasing sequences
- on levels of $\lambda^{<\lambda}$, we have almost disjoint families
- we have towers along branches (so no elements intersect the matrix)
- we have mad families on levels

  Maximality: *if $b \subseteq \omega$ is a challenge, use a complete subforcing to capture b; then find a $\sigma \in \lambda^{<\lambda}$ which is "outside" the complete subforcing, and use genericity to show that b and a_σ do have infinite intersection.*

Showing $\omega_1 = \mathfrak{h} = \mathfrak{b}$

Main ideas of the proof:

- recall that $\omega_1 \leq \mathfrak{h} \leq \mathfrak{b}$
- so it is enough to show that $\mathfrak{b} = \omega_1$
- we show that the ground model reals $\mathcal{B} := \omega^\omega \cap V_0$ remain unbounded

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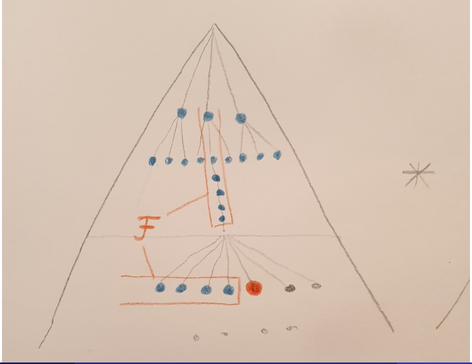

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- we represent our iteration as a "finer" iteration whose iterands are equivalent to Mathias forcing $\mathbb{M}(\mathcal{F})$ with respect to a filter $\mathcal{F}$
- we show that the filters $\mathcal{F}$ are $\mathcal{B}$-Canjar
    - a filter $\mathcal{F}$ is $\mathcal{B}$-Canjar, if $\mathbb{M}(\mathcal{F})$ preserves the unboundedness of $\mathcal{B}$
    - we use a combinatorial characterization by Guzmán-Hrušák-Martínez (extending a characterization of Canjarness given by Hrušák-Minami)
- at limit, unboundedness of $\mathcal{B}$ is preserved by theorem of Judah-Shelah

Thank you for your attention and enjoy Vienna...

Augarten, 3rd December 2020

Thank you for your attention and enjoy Vienna. . .

Stephansplatz (first district) during first lockdown, 9th April 2020

Thank you for your attention and enjoy Vienna. . .

Graben (first district) during first lockdown, 9th April 2020

Thank you for your attention and enjoy Vienna...

Old KGRC (Josephinum) during first lockdown, 9th April 2020

A filter $\mathcal{F}$ on ω is Canjar if $\mathbb{M}(\mathcal{F})$ does not add a dominating real.

Definition

A filter $\mathcal{F}$ on ω is $\mathcal{B}$-Canjar if $\mathbb{M}(\mathcal{F})$ preserves the unboundedness of $\mathcal{B}$.

Let X be a collection of finite subsets of ω. We say that

$$X \in (\mathcal{F}^{<\omega})^+ :\Longleftrightarrow \forall A \in \mathcal{F}\ \exists s \in X\ (s \subseteq A).$$

Theorem (Hrušák-Minami)

A filter $\mathcal{F}$ on ω is Canjar if and only if the following holds:
whenever $X_n \in (\mathcal{F}^{<\omega})^+$ for each $n \in \omega$, there exists an $f \in \omega^\omega$ such that

$$\bigcup_{n\in\omega} X_n \cap \mathcal{P}(f(n)) \in (\mathcal{F}^{<\omega})^+.$$

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