Asymptotic differential algebra and logarithmic transseries

Allen Gehret

Universität Wien, Institut für Mathematik,
Kurt Gödel Research Center

allen.gehret@univie.ac.at

KGRC Research Seminar
4. März 2020

du Bois-Reymond (1871–1882) "calculus of infinities"
Borel (1899)
Hahn (1907)
Hausdorff (1909, 1914)
Hardy (1911–1924) "Orders of infinity", log-exp functions
Rosenlicht (1972–1995)
Ecalle (1975–1993) Dulac Problem
Bourbaki (1976) Functions of a real variable, Hardy field appendix
Boshernitzan (1981–1987)
$\mathbb{T}$ van der Hoeven (1997–present)
Aschenbrenner (2000–present)
van den Dries (2000–present)
} Asymptotic Differential Algebra and Model Theory of Transseries

Discrete | **Continuous**

Geometric series $(r>0)$

$\sum r^n \begin{cases} \text{converges if } r<1 \\ \text{diverges if } r\geq 1 \end{cases}$ $\int^{\infty} r^x dx \begin{cases} \text{converges if } r<1 \\ \text{diverges if } r\geq 1 \end{cases}$

This plus comparison test can determine any function "visible at the exponential level" but cannot determine functions "visible at the polynomial level"

since $(1-\varepsilon)^n < \frac{1}{n^\alpha} < 1$ $(\alpha>0)$ as $n\to+\infty$

But we can convert polynomial level functions to the exponential level with

Cauchy Condensation Test (a_n) positive decreasing

$$\sum a_n \text{ converges} \iff \sum 2^n a_{2^n} \text{ converges}$$

E.g. $\sum \frac{1}{n^{1+\varepsilon}}$ converges $\iff \sum \frac{2^n}{(2^n)^{1+\varepsilon}}$ converges

$$\iff \sum \left(\frac{1}{2^\varepsilon}\right)^n \text{ converges}$$

$$\iff \varepsilon > 0$$

Can repeat this to get the "logarithmic criterion of order 1"

$\sum \frac{1}{n \ln(n)^{1+\varepsilon}}$ converges iff $\varepsilon > 0$

$\int^{\infty} \frac{dx}{x(\ln x)^{1+\varepsilon}}$ converges iff $\varepsilon > 0$

and more generally:

$\sum \frac{1}{l_0 l_1 \cdots l_{n-1} l_n^{1+\varepsilon}}$ converges iff $\varepsilon > 0$

$\int^{\infty} \frac{dx}{l_0 l_1 \cdots l_{n-1} l_n^{1+\varepsilon}}$ converges iff $\varepsilon > 0$

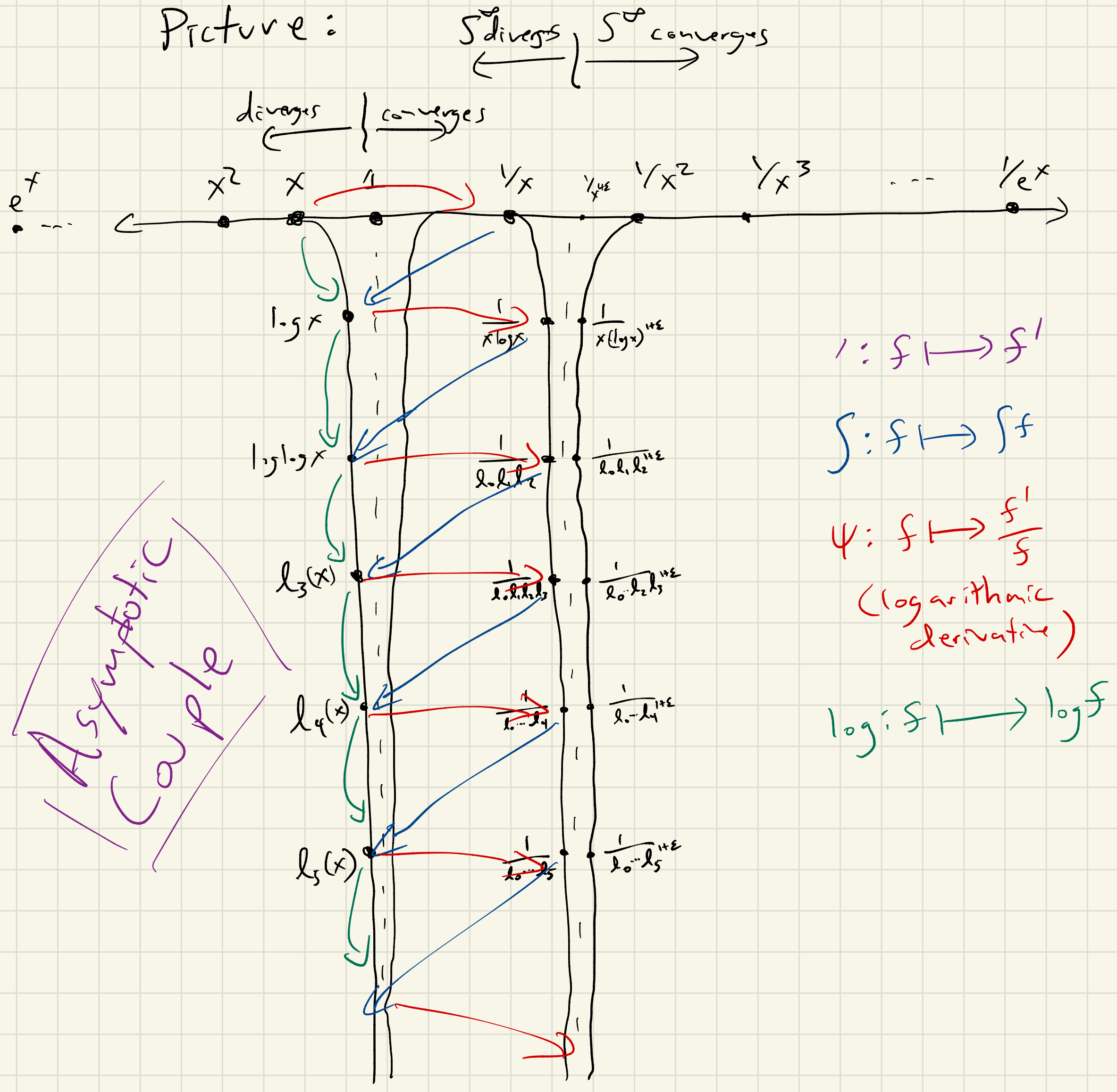
where $l_0 = x$, $l_1 = \ln x$, $l_{k+1} = \ln(l_k)$.

To summarize: $(\varepsilon > 0)$

$\int^{\infty}$ diverges	$\int^{\infty}$ converges
$\frac{1}{x}$	$\frac{1}{x^{1+\varepsilon}}$
$\frac{1}{x \ln x}$	$\frac{1}{x (\ln x)^{1+\varepsilon}}$
$\vdots$	$\vdots$

$$\frac{1}{x\,\ell_1 \ell_2 \cdots \ell_{n-1} \ell_n} \qquad \Bigg| \qquad \frac{1}{\ell_0 \ell_1 \cdots \ell_{n-1} \ell_n^{1+\varepsilon}}$$

Picture:



Hahn fields (or generalized power series): a construction

- Let a field C and an ordered (multiplicative) abelian group of "monomials" $\mathfrak{M} = (\mathfrak{M}; \cdot, \prec)$ be given.
- A set $\mathfrak{G} \subseteq \mathfrak{M}$ is **well-based** if there is no strictly increasing sequence $\mathfrak{m}_0 \prec \mathfrak{m}_1 \prec \mathfrak{m}_2 \prec \cdots$ in $\mathfrak{G}$.
- Given a function $f : \mathfrak{M} \to C$, written as a formal series $\sum_{\mathfrak{m} \in \mathfrak{M}} f_{\mathfrak{m}} \mathfrak{m}$ with $f_{\mathfrak{m}} := f(\mathfrak{m})$, the **support** of f is $\operatorname{supp} f := \{\mathfrak{m} \in \mathfrak{M} : f_{\mathfrak{m}} \neq 0\}$.
- The **Hahn field** $C[[\mathfrak{M}]] := \{f : \mathfrak{M} \to C : \operatorname{supp} f \text{ is well-based}\}$ is a valued field with pointwise addition and "series multiplication" (Neumann's lemma) with residue field C. Value group is an additive copy of $\mathfrak{M}$ with reverse ordering.
- Example: $\mathbb{C}[[t^{\mathbb{Z}}]]$, where $t := x^{-1}$, is the same as usual field of Laurent series $\mathbb{C}((x))$.

The (Ordered) Valued Field $\mathbb{T}_{\log}$

Definition (The valued field $\mathbb{T}_{\log}$ of logarithmic transseries)

$$\mathbb{T}_{\log} := \bigcup_n \mathbb{R}[[\mathfrak{L}_n]] \quad \text{union of spherically complete Hahn fields}$$

where $\mathfrak{L}_n$ is the *ordered group of logarithmic transmonomials*:

$$\mathfrak{L}_n := \ell_0^{\mathbb{R}} \cdots \ell_n^{\mathbb{R}} = \{\ell_0^{r_0} \cdots \ell_n^{r_n} : r_i \in \mathbb{R}\}, \quad \ell_0 = x, \ell_{m+1} = \log \ell_m$$

ordered such that $\ell_i \succ \ell_{i+1}^r \succ 1$ for all $r \in \mathbb{R}^{>0}$, $i = 0, \ldots, n-1$.

Typical elements of $\mathbb{T}_{\log}$ look like:

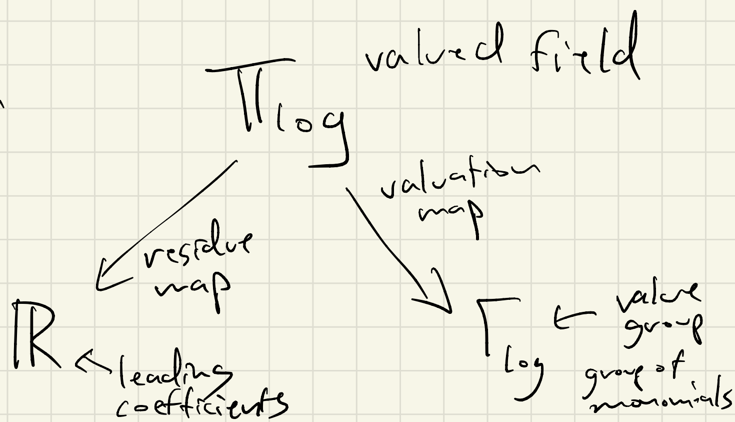
- $-2x^3 \log x + \sqrt{x} + 2 + \frac{1}{\log\log x} + \frac{1}{(\log\log x)^2} + \cdots$
- $\frac{1}{\log\log x} + \frac{1}{(\log\log x)^2} + \cdots + \frac{1}{\log x} + \frac{1}{(\log x)^2} + \cdots + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \cdots$

Note: $\mathbb{T}_{\log}$ is a real closed field and thus has a definable ordering.
Also: Residue field is $\mathbb{R}$ and value group $\Gamma_{\log}$ is additive copy of $\cup_n \mathfrak{L}_n$ with reverse ordering.

$$\frac{1}{x} + \frac{1}{x^{\log x}} + \frac{1}{\ell_0 \cdots \ell_2} + \frac{1}{\ell_0 \cdots \ell_3} + \cdots \notin \mathbb{T}_{\log}$$

$$\Gamma_{\log} \overset{!}{:=} \cup_n \mathfrak{L}_n$$

$\vee$



The derivation on $\mathbb{T}_{\log}$

$\mathbb{T}_{\log}$ comes equipped with the usual termwise derivative and logarithmic derivative:

$$\begin{aligned} f &\mapsto f' \\ f &\mapsto f^{\dagger} := f'/f, \quad (f \neq 0) \end{aligned}$$

subject to the usual rules: $\ell_0' = 1, \ell_1' = \ell_0^{-1}$, etc.
For example:

- $(x^3 \log x + \sqrt{x} + 2 + \cdots)' = 3x^2 \log x + x^2 + \frac{1}{2x^{1/2}} + \cdots$
- $\ell_n^{\dagger} = \frac{1}{\ell_0 \ell_1 \cdots \ell_n}$
- $(\frac{1}{\log\log x} + \frac{1}{(\log\log x)^2} + \cdots)' = -\frac{1}{x \log x (\log\log x)^2} - \frac{2}{x \log x (\log\log x)^3} + \cdots$
- $(\ell_0^{r_0} \cdots \ell_n^{r_n})^{\dagger} = r_0 \ell_0^{-1} + r_1 \ell_0^{-1} \ell_1^{-1} + \cdots + r_n \ell_0^{-1} \cdots \ell_n^{-1}$

This derivative makes $\mathbb{T}_{\log}$ into a differential field with field of constants $\mathbb{R}$.

H-fields: ordered valued differential fields with asymptotics

Definition

K an ordered valued differential field. We call K an H-**field** if

H1 for all $f \in K$, if $f > C$, then $f' > 0$;

H2 $\mathcal{O} = C + \mathcal{o}$ where $\mathcal{O} = \{g \in K : |g| \leq c \text{ for some } c \in C\}$ is the (convex) valuation ring of K and $\mathcal{o}$ is the maximal ideal of $\mathcal{O}$

Example

$\mathbb{T}_{\log}$ is an H-field, also any Hardy field containing $\mathbb{R}$ is an H-field.

Example

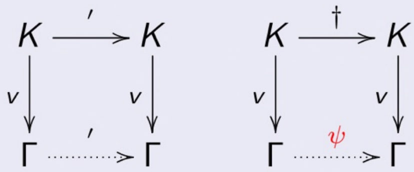
$\mathbb{T}$, the differential field of *logarithmic-exponential transseries* is naturally an H-field, and contains $\mathbb{T}_{\log}$. It is closed under exp. Typical element:

$$-3e^{e^x} + e^{\frac{e^x}{\log x} + \frac{e^x}{\log^2 x} + \frac{e^x}{\log^3 x} + \cdots} - x^{11} + 7 + \frac{\pi}{x} + \frac{1}{x \log x} + \cdots + e^{-x} + 2e^{-x^2} + \cdots$$

The asymptotic couple (Γ, ψ) of an H-field K

Fact

For $f \in K^\times$ such that $v(f) \neq 0$, the values $v(f')$ and $v(f^\dagger)$ depend only on $v(f)$.



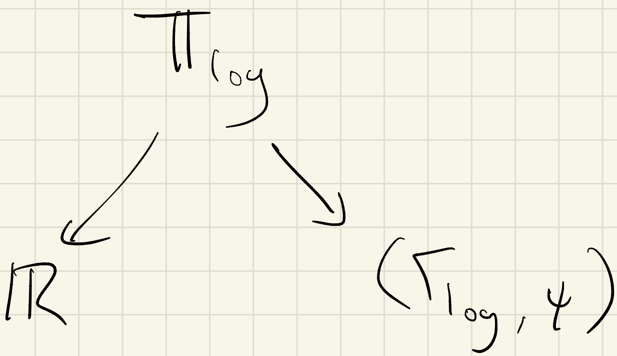
(Γ is the value group of K)

Definition (Rosenlicht)

The pair (Γ, ψ) is the *asymptotic couple of K*.

Theorem (G)

$\mathrm{Th}(\Gamma_{\log}, \psi)$, *the asymptotic couple of $\mathbb{T}_{\log}$, has QE in a natural language, is model complete, has NIP, and is distal (with Elliot Kaplan, 2018).*



H-fields: two technical properties

Both $\mathbb{T}$ and $\mathbb{T}_{\log}$ enjoy two additional (first-order) properties:

- **ω-free**: this is a very strong and robust property which prevents certain deviant behavior
  $\forall f \neq 0\ \exists g \not\prec 1 [g' \asymp f]\ \ \&\ \ \forall f\ \exists g \succ 1 [f + 2g^{\dagger\dagger'} + 2(g^{\dagger\dagger})^2 \not\succcurlyeq (g^{\dagger})^2]$
- **newtonian**: this is a variant of "differential-henselian"; it essentially means that you can simulate being differential henselian arbitrarily well by sufficient coarsenings and compositional conjugations ($\partial \mapsto \phi\partial$).

$\mathbb{T}_{\log}$ satisfies both of these properties because it has integration and is a union of spherically complete H-fields, each with a smallest "comparability class":

$$\mathbb{T}_{\log} := \bigcup_n \mathbb{R}[[\ell_0^{\mathbb{R}} \cdots \ell_n^{\mathbb{R}}]]$$

$\mathbb{T}$

|

$\mathbb{T}_{\log}$

~~$\mathbb{R}[[\bigcup_n \mathfrak{L}_n]]$~~

H-fields: integrals and exponential integrals

Another nice property:

Definition

We call a real closed H-field K **Liouville closed** if

$$K' = K \quad \text{and} \quad (K^{\times})^{\dagger} = K$$

$\mathbb{T}$ is Liouville closed, however...
$\mathbb{T}_{\log}$ is NOT Liouville closed:

$$(\mathbb{T}_{\log})' = \mathbb{T}_{\log} \quad \text{but} \quad (\mathbb{T}_{\log}^{\times})^{\dagger} \neq \mathbb{T}_{\log}$$

first place $\mathbb{T}$ and $\mathbb{T}_{\log}$ differ

E.g., an element f such that $f^{\dagger} = 1$ would have to behave like e^{x}..

The field $\mathbb{T}$: a success story

Let $\mathcal{L} = \{0, 1, +, -, \cdot, \partial, \leq, \preccurlyeq\}$
The following result is the starting point for the model theory of $\mathbb{T}_{\log}$:

Theorem (Aschenbrenner, van den Dries, van der Hoeven, 2015)

$\mathbb{T}$ is model complete as an $\mathcal{L}$-structure. Furthermore, $\mathrm{Th}_{\mathcal{L}}(\mathbb{T})$ is axiomatized by:

- *real closed, ω-free, newtonian, H-field such that* $\forall \epsilon \prec 1, \partial(\epsilon) \prec 1$;
- *Liouville closed*
  - $K' = K$
  - $(K^{\times})^{\dagger} = K$

Recall: a structure M is *model complete* if every definable subset of M^n is existentially definable (for every n). A starting point for model completeness of $\mathbb{T}_{\log}$ is to try to make both $(\mathbb{T}_{\log}^{\times})^{\dagger}$ and its complement existentially definable.

Investigating $(\mathbb{T}_{\log}^{\times})^{\dagger}$

$f \in (\mathbb{T}_{\log}^{\times})^{\dagger} \iff$ **there exists** $g \in \mathbb{T}_{\log}^{\times}$ such that $g^{\dagger} = f$

Given $f \in \mathbb{T}_{\log}^{\times}$, we can write it uniquely as

$$f = c\ell_0^{r_0} \cdots \ell_n^{r_n}(1+\epsilon) \quad \text{for some infinitesimal } \epsilon \prec 1 \text{ and some } c \in \mathbb{R}^{\times}$$

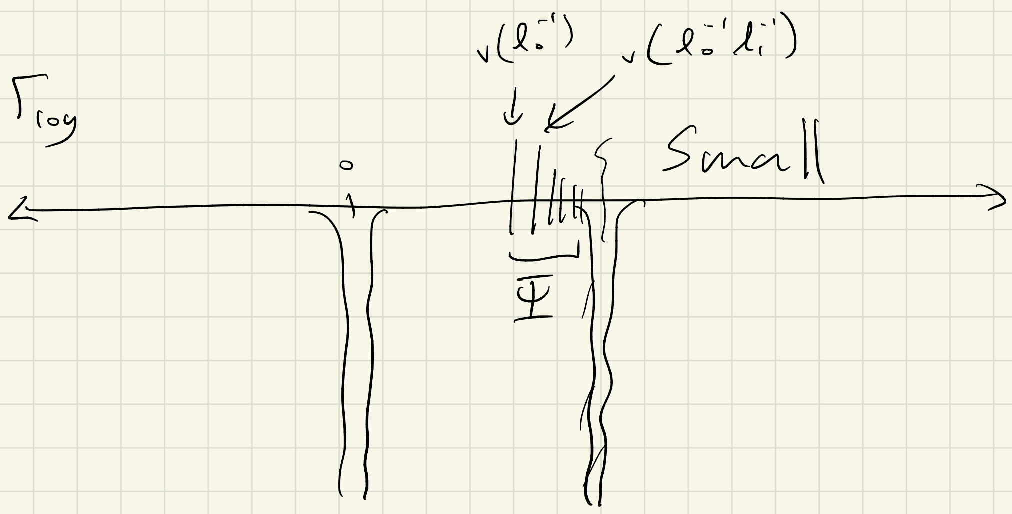
Then we compute the logarithmic derivative:

$$(c\ell_0^{r_0} \cdots \ell_n^{r_n}(1+\epsilon))^{\dagger} = r_0\ell_0^{-1} + r_1\ell_0^{-1}\ell_1^{-1} + \cdots + r_n\ell_0^{-1} \cdots \ell_n^{-1} + \underbrace{\frac{\epsilon'}{1+\epsilon}}_{\text{“small”}}$$

Note: $v(\ell_0^{-1} \cdots \ell_n^{-1}) \in \Psi := \psi(\Gamma_{\log}^{\neq})$ and $v(\epsilon'/(1+\epsilon)) > \Psi$.

Fact

$f \notin (\mathbb{T}_{\log}^{\times})^{\dagger} \iff$ there exists $g \in \mathbb{T}_{\log}^{\times}$ *such that* $v(f - g^{\dagger}) \in \Psi^{\downarrow} \setminus \Psi$



Introducing LD-H-fields

From now on all H-fields will have asymptotic integration ($\Gamma = (\Gamma^{\neq})'$).
Let K be an H-field and $\mathrm{LD} \subseteq K$.
We call the pair (K, LD) an LD-H-**field** if:

LD1 LD is a C_K-vector subspace of K;

LD2 $(K^\times)^\dagger \subseteq \mathrm{LD}$;

LD3 $\mathrm{I}(K) := \{y \in K : y \preccurlyeq f' \text{ for some } f \in \mathcal{O}\} \subseteq \mathrm{LD}$; and

LD4 $v(\mathrm{LD}) \subseteq \Psi \cup (\Gamma^>)' \cup \{\infty\}$.

We say an LD-H-field (K, LD) is **full** if:

(full) For every $a \in K \setminus \mathrm{LD}$, there is $b \in \mathrm{LD}$ such that $v(a-b) \in \Psi^{\downarrow} \setminus \Psi$,

and we say it is Ψ-**closed** if it is full and $\mathrm{LD} = (K^\times)^\dagger$.

Example

$(\mathbb{T}_{\log}, (\mathbb{T}_{\log}^\times)^\dagger)$ and $(\mathbb{T}, \mathbb{T})$ are both Ψ-closed LD-H-fields.

Model completeness conjecture for $\mathbb{T}_{\log}$

Let $\mathcal{L}_{\mathrm{LD}} := \{0, 1, +, -, \cdot, \partial, \leq, \preccurlyeq, \mathrm{LD}\}$ where LD is a unary relation symbol. Let $T_{\log}$ be the $\mathcal{L}_{\mathrm{LD}}$-theory whose models are precisely the LD-H-fields (K, LD) such that:

1. K is real closed, ω-free, and newtonian;
2. (K, LD) is Ψ-closed; and
3. $(\Gamma, \psi) \models \mathrm{Th}(\Gamma_{\log}, \psi)$, where (Γ, ψ) is the asymptotic couple of K.

Conjecture

The theory $T_{\log}$ is model complete.

Embedding version of conjecture

Let (K, LD) and (L, LD_1) be models of $T_{\log}$ and suppose (E, LD_0) is a full ω-free LD-H-subfield of (K, LD) such that $(\mathbb{Q}\Gamma_E, \psi) \models \mathrm{Th}(\Gamma_{\log}, \psi)$. Let $i : (E, \mathrm{LD}_0) \to (L, \mathrm{LD}_1)$ be an embedding of LD-H-fields. Assume (L, LD_1) is $|K|^+$-saturated and (K, LD) is $\aleph_0$-saturated. Then i extends to an embedding $(K, \mathrm{LD}) \to (L, \mathrm{LD}_1)$ of LD-H-fields.

Algebraic Extensions of LD-H-fields

Given LD-H-fields (K, LD) and (L, LD^*) such that $K \subseteq L$, we say that (L, LD^*) **is an extension of** (K, LD) (notation $(K, \mathrm{LD}) \subseteq (L, \mathrm{LD}^*)$) is $\mathrm{LD}^* \cap K = \mathrm{LD}$.

Proposition

Suppose L is an algebraic extension of K, (K, LD) **is full**, and $(\Gamma, \psi) \models \mathrm{Th}(\Gamma_{\log}, \psi)$. Then there is a **unique** LD-set $\mathrm{LD}^* \subseteq L$ such that $(K, \mathrm{LD}) \subseteq (L, \mathrm{LD}^*)$; equipped with this LD-set, (L, LD^*) **also is full**. Important case: L is a real closure of K.

Constant Field Extensions of LD-H-fields

Suppose $K \subseteq L$ is an extension of H-fields such that $L = K(C_L)$, so L is a constant field extension of K.

Proposition

Suppose K is henselian, $(\Gamma, \psi) \models \mathrm{Th}(\Gamma_{\log}, \psi)$, and (K, LD) **is full**. Then there is a **unique** LD-set $\mathrm{LD}^* \subseteq L$ such that $(K, \mathrm{LD}) \subseteq (L, \mathrm{LD}^*)$; equipped with this LD-set, (L, LD^*) **also is full**.

Thus adding new constants will never be an issue!

The Ψ-closure of an LD-H-field

Definition

We say an LD-H-field extension $(K^{\Psi}, \mathrm{LD}^{\Psi})$ of (K, LD) is a **Ψ-closure of** (K, LD) if K^{Ψ} is real closed, $(K^{\Psi}, \mathrm{LD}^{\Psi})$ is Ψ-closed, and for any LD-H-field extension (L, LD^{*}) of (K, LD) such that L is real closed and (L, LD^{*}) is Ψ-closed, there is an embedding $(K^{\Psi}, \mathrm{LD}^{\Psi}) \to (L, \mathrm{LD}^{*})$ of LD-H-fields over (K, LD).

Theorem

Suppose (K, LD) *is full, is* λ*-free, and* $(\Gamma, \psi) \models \mathrm{Th}(\Gamma_{\log}, \psi)$. *Then* (K, LD) *has a unique* Ψ*-closure. This* Ψ*-closure will be differentially-algebraic over* K*, does not contain any proper real closed and* Ψ*-closed differential subfields containing* K*, and its asymptotic couple will model* $\mathrm{Th}(\Gamma_{\log}, \psi)$.

Newtonization: a reduction to the linear case

Suppose K is ω-free, $(\Gamma, \psi) \models \mathrm{Th}(\Gamma_{\log}, \psi)$ and let K^{nt} be the *newtonization* of K (a newtonian extension of K with a natural universal property).

What we would like to prove:

Suppose (K, LD) is full. Then $\mathrm{LD}^{\mathrm{nt}} := \mathrm{LD} + \mathrm{I}(K^{\mathrm{nt}})$ is the unique LD-set on K^{nt} such that $(K, \mathrm{LD}) \subseteq (K^{\mathrm{nt}}, \mathrm{LD}^{\mathrm{nt}})$; equipped with this LD-set, $(K^{\mathrm{nt}}, \mathrm{LD}^{\mathrm{nt}})$ also is full.

It suffices(!!!) to prove the linear case:

Conjecture 1 (Linear newtonian conjecture)

There is a linearly newtonian H-field L such that $K \subseteq L \subseteq K^{\mathrm{nt}}$ and $\mathrm{LD}^* := \mathrm{LD} + \mathrm{I}(L)$ is the unique LD-set on L such that $(K, \mathrm{LD}) \subseteq (L, \mathrm{LD}^*)$; equipped with this LD-set, (L, LD^*) also is full.

Linearly newtonian is the fragment of *newtonian* that only involves degree 1 differential polynomials (differential operators).

The other case we need to handle

Conjecture 2 (Immediate differentially-transcendental conjecture)

Suppose (L, LD^*) is an LD-H-field extension of (K, LD) such that $(K, \mathrm{LD}), (L, \mathrm{LD}^*) \models T_{\log}$, and suppose there is $y \in L \setminus K$ such that $K\langle y \rangle$ is an immediate extension of K (so y is necessarily differentially transcendental over K since K is asymptotically d-algebraically maximal). Then $\mathrm{LD}_y := \mathrm{LD} + \mathrm{I}(K\langle y \rangle)$ is the unique LD-set on $K\langle y \rangle$ such that $(K, \mathrm{LD}) \subseteq (K\langle y \rangle, \mathrm{LD}_y)$; equipped with this LD-set, $(K\langle y \rangle, \mathrm{LD}_y)$ also is full.

Note: there are weaker versions of Conjectures 1 and 2 which will suffice for our purposes (in case as written they are false).

The main "result"

Theorem (G)

Assume Conjectures 1 and 2 hold. Then $\mathbb{T}_{\log}$ is model complete as an LD-H-field.

Constants and solutions of ODEs

Recall from calculus/ODEs:

- The differential equation

$$Y' - \cos t \ = \ 0$$

  has solutions

$$\{\sin t + c_0 : c_0 \in \mathbb{R}\}$$

- The differential equation

$$Y'' - 3Y' + 2Y \ = \ 0$$

  has solutions

$$\left\{c_0 \, e^{2t} + c_1 \, e^{t} : c_0, c_1 \in \mathbb{R}\right\}$$

So the solutions are "controlled" by the constant field $\mathbb{R}$

Co-analyzability: a form of "controlling"

Suppose K is ω-saturated, $C \subseteq K$ is a definable set.

Definition

Let $S \subseteq K^n$ be definable. For $r \in \mathbb{N}$ we say S **is co-analyzable in** r **steps (relative to** K **and** C**)** if:

(C_0) S is co-analyzable in 0 steps iff S is finite;

(C_{r+1}) S is co-analyzable in $r+1$ steps iff for some definable set $R \subseteq C \times K^n$,

1. the natural projection $C \times K^n \to K^n$ maps R onto S;
2. for each $c \in C$, the section $R(c) := \{s \in K^n : (c,s) \in R\}$ above c is co-analyzable in r steps.

We call S **co-analyzable** if S is co-analyzable in r steps for some r.

Consequences of co-analyzability

Fact

Suppose $\mathcal{L}$ is countable, T is complete $\mathcal{L}$-theory such that $T \vdash \exists x C(x)$. Then the following are equivalent for a formula $\varphi(x)$:

1. *For some $K \models T$, $\varphi(K)$ is co-analyzable (relative to C),*
2. *For every $K \models T$, $\varphi(K)$ is co-analyzable,*
3. *For every $K \models T$, if C_K is countable, then so is $\varphi(K)$,*
4. *For all $K \preccurlyeq K^* \models T$, if $C_K = C_{K^*}$, then $\varphi(K) = \varphi(K^*)$.*

Moral: (3) and (4) show there is some (possibly complicated) relationship between C and the definable set $\varphi(K)$.

Co-analyzability and differential equations

Theorem (G)

Suppose K is an H-field such that

1. *K is real closed, ω-free, and newtonian, and*
2. *K is Ψ-closed.*

Then for every nonzero differential polynomial $P \in K\{Y\}$, the set

$$Z(P) \;:=\; \{y \in K : P(y) = 0\}$$

is co-analyzable relative to the constant field C.

This was known for $\mathbb{T}$ (2016, ADH), but new for $\mathbb{T}_{\log}$ and other H-fields.