DVR 0065528

Programme on

"Optimal Transport"

April 15 – June 14, 2019

organized by

Mathias Beiglböck (U Vienna), Alessio Figalli (ETH Zurich), Jan Maas (IST Austria), Robert McCann (U Toronto), Justin Solomon (MIT, Boston)

Workshop 1
Optimal Transport: from Geometry to Numerics

May 13 – 17, 2019

- **Monday, May 13, 2019**

  14:00 – 14:45 **Yann Brenier**
  Various formulations of the Muskat model

  14:45 – 15:30 **Cornelia Mihaila**
  Bubbling with L^2 almost constant mean curvature and an Alexandrov type theorem for crystals

  15:30 – 16:00 *Coffee / Tea break*

  16:00 – 16:45 **Mikaela Iacobelli**
  From quantization of measures to weighted ultrafast diffusion equations

  17:00 – 19:00 *Reception*

- **Tuesday, May 14, 2019**

  09:15 – 10:00 **Gabriel Peyré**
  Optimal Transport for Machine Learning

  10:00 – 10:45 **Nicolás García Trillos**
  Large sample asymptotics of spectra of Laplacians and semilinear elliptic PDEs on random geometric graphs

  10:45 – 11:15 *Coffee / Tea break*

  11:15 – 12:00 **Marco Cuturi**
  On the several ways to regularize optimal transport

- **Wednesday, May 15, 2019**

  09:15 – 10:00 **Andrea Mondino**
  Optimal transport and quantitative geometric inequalities

  10:00 – 10:45 **Zoltan Balogh**
  Geometric inequalities via OMT

  10:45 – 11:15 *Coffee / Tea break*

11:15 – 12:00 **Martin Huesmann**
Optimal matching and quantitative linearization results for Monge-Ampere equations

12:00 – 12:45 **Martin Rumpf**
Computation of Optimal Transport on Discrete Metric Measure Spaces

- **Thursday, May 16, 2019**

  09:15 – 10:00 **Paola Gori-Giorgi**
  Beyond the optimal transport limit of density functional theory: dispersion interactions with fixed marginals

  10:00 – 10:45 **Virginie Ehrlacher**
  Marginal constrained optimal transport problem

  10:45 – 11:15 *Coffee / Tea break*

  11:15 – 12:00 **Max von Renesse**
  Dean-Kawasaki Dynamics: Particle-ular solutions for an Ill-Posed SPDE

  12:00 – 14:00 *Lunch break*

  14:00 – 14:45 **Dario Trevisan**
  An infinite dimensional Gaussian random matching problem

  14:45 – 15:30 **Carola-Bibiane Schönlieb**
  Wasserstein for learning image regularisers

  15:30 – 16:00 *Coffee / Tea break*

  16:00 – 16:45 **Bruno Lévy**
  Incompressible fluid simulation with semi-discrete optimal transport

- **Friday, May 17, 2019**

  09:15 – 10:00 **Alexander Mielke**
  Transport versus growth and decay: the (spherical) Hellinger-Kantorovich distance between arbitrary measures

  10:00 – 10:45 **Daniel Matthes**
  Variational approximations of Wasserstein gradient flows (beyond minimizing movements)

  10:45 – 11:15 *Coffee / Tea break*

  11:15 – 12:00 **Giuseppe Buttazzo**
  Optimal transport between mutually singular measures

All talks take place at ESI, Boltzmann Lecture Hall.

Abstracts

Zoltan Balogh (University of Bern)
Geometric inequalities via OMT

Applications of OMT in proving geometric inequalities of the type Borel-Brascamp-Lieb are considered in various spaces including Euclidean, Riemannian and sub-Riemannian settings. The cases of equality are also considered. This is joint work with Alexandru Kristaly and Kinga Sipos.

Yann Brenier (CNRS, DMA-ENS Paris)
Various formulations of the Muskat model

The Muskat model, which describes the filtration of an incompressible fluid in a porous medium, has played a crucial role in optimal transport theory since the work of Otto concerning its gradient flow structure in the mid 90s. In the last 10 years it has become an important example in the theory of convex integration applied to fluid equations, after Cordoba, Faraco, Szekelyhidi... I will first recall how a very simple and computationally effective time discrete version of the Muskat model can be designed in terms of polar factorization of maps. Then I will discuss various possible formulations (multiphasic, dissipative...) of this model.

Giuseppe Buttazzo (Università degli Studi di Pisa)
Optimal transport between mutually singular measures

We study the Wasserstein distance between two measures μ, ν which are mutually singular. In particular, we are interested in minimization problems of the form

$$W(\mu, \mathcal{A}) = \inf \big\{ W(\mu, \nu) \ : \ \nu \in \mathcal{A} \big\}$$

where μ is a given probability and $\mathcal{A}$ is contained in the class $\mu^{\perp}$ of probabilities that are singular with respect to μ. Several cases for $\mathcal{A}$ are considered; in particular, when $\mathcal{A}$ consists of L^1 densities bounded by a constant, the optimal solution is given by the characteristic function of a domain. Some regularity properties of these optimal domains are also studied. Some numerical simulations are included, as well as the double minimization problem

$$\min \big\{ P(B) + kW(A, B) \ : \ |A \cap B| = 0, \ |A| = |B| = 1 \big\},$$

where $k > 0$ is a fixed constant, $P(A)$ is the perimeter of A, and both sets A, B may vary.

Marco Cuturi (Google Brain & ENSAE)
On the several ways to regularize optimal transport

I will show in this talk how regularization, either explicitly carried out as a statistical procedure or implicitly carried out and presented as a computational trick, is fundamental for optimal transport to work in applications to data sciences. I will present two such regularizations, either by regularizing the transport plan by entropy, or by projecting measures on maximally informative subspaces. (presentations based on joint works with G. Peyré, A. Genevay, F. Bach and F.P. Paty)

Virginie Ehrlacher (CERMICS, ENPC)
Marginal constrained optimal transport problem

The aim of this talk is to present recent results a relaxation of multi-marginal optimal transport problems with a view to the design of numerical schemes to approximate the exact optimal transport problem. More precisely, the approximate problem considered in this talk consists in relaxing the marginal constraints into a finite number of moments constraints. Using Tchakhaloff's theorem, it is possible to prove the existence of minimizers of this relaxed problem and characterize them as discrete measures charging a number of points which scales at most linearly with the number of marginals in the problem. This result opens the way to the design of new numerical schemes exploiting the structure of these minimizers, and preliminary numerical results will be presented.

This is joint work with A. Alfonsi, R. Coyaud and D. Lombardi.

Nicolás García Trillos (University of Wisconsin-Madison)
Large sample asymptotics of spectra of Laplacians and semilinear elliptic PDEs on random geometric graphs

Given a data set $\mathcal{X} = \{x_1, \ldots, x_n\}$ and a weighted graph structure $\Gamma = (\mathcal{X}, W)$ on $\mathcal{X}$, graph based methods for learning use analytical notions like graph Laplacians, graph cuts, and Sobolev semi-norms to formulate optimization problems whose solutions serve as sensible approaches to machine learning tasks. When the data set consists of samples from a distribution supported on a manifold (or at least approximately so), and the weights depend inversely on the distance between the points, a natural question to study concerns the behavior of those optimization problems as the number of samples goes to infinity. In this talk I will focus on optimization problems closely connected to clustering and supervised regression that involve the graph Laplacian. For clustering, the spectrum of the graph Laplacian is the fundamental object used in the popular spectral clustering algorithm. For regression, the solution to a semilinear elliptic PDE on the graph provides the minimizer of an energy balancing regularization and data fidelity, a sensible object to use in non-parametric regression.

Using tools from optimal transport, calculus of variations, and analysis of PDEs, I will discuss a series of results establishing the asymptotic consistency (with rates of convergence) of many of these analytical objects, as well as provide some perspectives on future research directions.

Paola Gori-Giorgi (VU Amsterdam)
Beyond the optimal transport limit of density functional theory: dispersion interactions with fixed marginals

The optimal transport (OT) limit of the density functional theory (DFT) formulation of the many-electron problem has been the object of several studies in the last decade with many new results. In this talk I will outline some strategies to include the quantum effects due to kinetic energy that are missed by the strict OT limit. In particular, I will present a new approach to dispersion (a.k.a. van der Waals) interactions in which the diagonal of the density matrix (marginal) of the two interacting fragments is kept fixed. This defines a modified OT problem in which an anisotropic harmonic cost is corrected with a gradient term. Although the approach is not exact by construction (as in the exact case the marginals are known to change) it highly simplifies the problem and yields surprisingly accurate (and in some cases even exact) results.

Martin Huesmann (University of Vienna)
Optimal matching and quantitative linearization results for Monge-Ampere equations

Motivated by applications to optimal matching problems we develop a large scale regularity theory for the Monge-Ampere equation with potentially rough data. We show that given a measure μ which is assumed to be close to the Lebesgue measure in Wasserstein distance at all scales, then the displacement of the macroscopic optimal coupling is quantitatively close at all scales to the gradient of the solution of the corresponding Poisson equation. The main ingredient we use is an harmonic approximation result for the optimal transport plan between arbitrary measures. This is used in a Campanato iteration which transfers the information through the scales. (based on joint work with Michael Goldman and Felix Otto)

Mikaela Iacobelli (Durham University)
From quantization of measures to weighted ultrafast diffusion equations

In this talk I will discuss some recent results on the asymptotic behaviour of a family of weighted ultrafast diffusion PDEs. These equations are motivated by the gradient flow approach to the problem of quantization of measures, introduced in a series of joint papers with Emanuele Caglioti and François Golse. In this presentation I will focus on a recent paper with Francesco Saverio Patacchini and Filippo Santambrogio, where we use the JKO scheme to obtain existence, uniqueness, and exponential convergence to equilibrium under minimal assumptions on the data.

Bruno Lévy (INRIA)
Incompressible fluid simulation with semi-discrete optimal transport

I will present algorithms to simulate incompressible fluids based on optimal transport. The Gallouet-Merigot scheme decomposes the fluid into a set of cells, and controls the volume of these cells. The decomposition is parameterized by the positions of a set of points, and the cells are the pre-images of these points through the transport (Laguerre diagram). I will explain how to compute such a Laguerre diagram in various configurations (interior of an arbitrary polyhedron, intersection between Laguerre diagram and spheres, periodic boundary conditions) and show some applications (simulation of fluids with free surface,viscosity and surface tension). I will also show some applications in astrophysics (with R. Mohayahee, J.-M. Alimi and Q. Merigot).

Daniel Matthes (TU Munich)
Variational approximations of Wasserstein gradient flows (beyond minimizing movements)

For the construction of gradient flows in metric spaces like Wasserstein, approximation via the time discrete implicit Euler scheme (a.k.a. minimizing movements or JKO method) is certainly the most common choice. We shall discuss two alternative approximation methods, which are variational as well, but have different advantages and disadvantages.

Our first approach is an adaptation of the BDF2 method from ODE theory. We prove that under a certain convexity hypotheses on the potential of the flow, the time-discrete approximation converges to an EVI-solution. In the context of Wasserstein, all displacement semi-convex functionals fall into that class. Numerical experiments indicate that the order of convergence is two, as expected.

The second method, sometimes called weighted energy dissipation, uses a global elliptic-in-time regularization of the gradient flow. For particular (also non-convex) Wasserstein gradient flows, we prove convergence in the limit of vanishing regularization to a weak solution of the respective PDE. Our approach is tailored to these applications and uses only little of the recently developed general abstract theory.

Joint work with Simon Plazotta, Giuseppe Savare, and Stefano Lisini.

Alexander Mielke (WIAS Berlin)
Transport versus growth and decay: the (spherical) Hellinger-Kantorovich distance between arbitrary measures

The Hellinger-Kantorovich distance can be seen as an interpolation (or inf-convolution) between the Wasserstein-Kantorovich distances of optimal transport and the Hellinger-Kakutani distance. The latter is defined between measures of arbitrary mass. We provide several useful characterization of these distances and discuss geometric features involving cone constructions, where we interprete the space of all measures as a cone over the probability measures.

(This is joint work with Matthias Liero, Giuseppe Savare, and Vaios Laschos.)

Cornelia Mihaila (University of Chicago)
Bubbling with L^2 almost constant mean curvature and an Alexandrov type theorem for crystals

I will discuss a recent result in which an Alexandrov-type theorem for L^2 almost constant anisotropic mean curvature sets is proven. In addition I will provide a description of critical points/local minimizers for elliptic energies interacting with a confinement potential. An improvement on previous almost constant mean curvature results is our use of L^2 versus C^0 closeness, since this should have applications in mean curvature flow and is new even in the isotropic case. This talk is based on a joint work with Matias Delgadino, Francesco Maggi, and Robin Neumayer.

Andrea Mondino (University of Warwick)
Optimal transport and quantitative geometric inequalities

The goal of the talk is to discuss a quantitative version of the Levy-Gromov isoperimetric inequality (joint with Cavalletti and Maggi) as well as other geometric/functional inequalities (joint with Cavalletti and Semola). Given a closed Riemannian manifold with strictly positive Ricci tensor, one estimates the measure of the symmetric difference of a set with a metric ball with the deficit in the Levy-Gromov inequality. The results are obtained via a quantitative analysis based on the localisation method via L^1-optimal transport.

Gabriel Peyré (CNRS and Ecole Normale Supérieure Paris)
Optimal Transport for Machine Learning

Optimal transport (OT) has become a fundamental mathematical tool at the interface between calculus of variations, partial differential equations and probability. It took however much more time for this notion to become mainstream in numerical applications. This situation is in large part due to the high computational cost of the underlying optimization problems. There is a recent wave of activity on the use of OT-related methods in fields as diverse as image processing, computer vision, computer graphics, statistical inference, machine learning. In this talk, I will review an emerging class of numerical approaches for the approximate resolution of OT-based optimization problems. This offers a new perspective for the application of OT in high dimension, to solve supervised (learning with transportation loss function) and unsupervised (generative network training) machine learning problems. More information and references can be found on the website of our book Computational Optimal Transport" https://optimaltransport.github.io/

Max von Renesse (University of Leipzig)
Dean-Kawasaki Dynamics: Particle-ular solutions for an Ill-Posed SPDE

The Dean-Kawasaki Equation is well-known SPDE model from physics which arises in macroscopic fluctuation theory and glassy materials. Mathematically it can be understood as a stochastic heat equation with a singular multiplicative noise structure that is formally associated to a Brownian motion on the Wasserstein space of probability measures equipped the the optimal transportation distance. We show that this model is generally ill posed except in certain regimes when the solution is trivial. We also present some modifications of the equation which admit non trivial solutions in terms of coalescing-fragmentating particle systems which exhibit glassy behaviour. Finally we show that the short time asymptotics are indeed governed by the quadratic Wasserstein distance of optimal transportation.

Joint work with Vitalii Konarovskyi and Tobias Lehmann (Leipzig)

Martin Rumpf (University of Bonn)
Computation of Optimal Transport on Discrete Metric Measure Spaces

In this talk we investigate the numerical approximation of an analogue of the Wasserstein distance for optimal transport on graphs that is defined via a discrete modification of the Benamou–Brenier formula. This approach involves the logarithmic mean of measure densities on adjacent nodes of the graph. For this model a variational time discretization of the probability densities on graph nodes and the momenta on graph edges is proposed. A robust descent algorithm for the action functional is derived, which in particular uses a proximal splitting with an edgewise nonlinear projection on the convex subgraph of the logarithmic mean. Thereby, suitable chosen slack variables avoid a global coupling of probability densities on all graph nodes in the projection step. For the time discrete action functional Γ–convergence to the time continuous action is established. Numerical results for a selection of test cases show qualitative and quantitative properties of the optimal transport on graphs.

This is joint work with Matthias Erbar, Bernhard Schmitzer, and Stefan Simon.

Carola-Bibiane Schönlieb (University of Cambridge)
Wasserstein for learning image regularisers

In this talk I will present the results of our recent NeurIPS paper on adversarial regularisers for inverse problems which are trained with an approximate Wasserstein loss. I will discuss what we understand about this training procedure theoretically and showcase the performance of the learned regulariser for computed tomography imaging.

This is joint work with Sebastian Lunz and Ozan Öktem.

Dario Trevisan Università degli Studi di Pisa
An infinite dimensional Gaussian random matching problem

We investigate upper and lower bounds for the Wasserstein-Kantorovich distance between random empirical measures and the common law for a sequence of i.i.d. infinite dimensional Gaussian random variables taking values in a Hilbert space. The technique uses random PDE's and semigroups in Gaussian Hilbert spaces. As an application, we obtain quantitative rates of convergence for samples of Brownian paths to the Wiener measure.

Joint work with E. Stepanov.