Pfaffian Orientations and K_4-free Graphs

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Valiant (1979) showed that, unless P = NP, there is no polynomial-time algorithm to compute the number of perfect matchings of a given graph — even if the input graph is bipartite. Earlier, Kasteleyn (1963) introduced the notion of a *Pfaffian orientation*, and showed that if a graph admits such an orientation then the number of its perfect matchings may be obtained by computing the determinant of the associated skew-symmetric matrix. A graph is Pfaffian if it admits a Pfaffian orientation. Kasteleyn proved that every planar graph is Pfaffian, and thus solved the dimer problem in statistical mechanics. The smallest non-Pfaffian graph is $K_{3,3}$.

Special types of minors, known as *conformal minors*, play a key role in the theory of Pfaffian graphs — analogous to the role of *topological minors* in the theory of planar graphs. Little (1975) proved that a bipartite graph G is Pfaffian if and only if G does not contain $K_{3,3}$ as a conformal minor (or, in other words, if and only if G is $K_{3,3}$-free). Several years later, the groundbreaking works of Robertson, Seymour and Thomas (1999), and independently of McCuaig (2004), led to a structure theorem for Pfaffian bipartite graphs, and thus to a polynomial-time algorithm for deciding whether a bipartite graph is Pfaffian.

Fischer and Little (2001) proved that a near-bipartite graph is Pfaffian if and only if it does not contain any of seven specific graphs as a conformal minor — these graphs are $K_{3,3}$, Cubeplex, Twinplex, and four other graphs derived from these (by replacing a specified vertex by a triangle). Near-bipartite graphs are an important class of (nonbipartite) graphs that arise naturally from the ear-decomposition theory of Lovász and Plummer.

Norine and Thomas (2008) showed that, unlike the bipartite and near-bipartite cases, it is not possible to characterize all Pfaffian graphs by excluding a finite number of graphs as conformal minors. In light of their work, it would be interesting to even identify rich classes of Pfaffian graphs (that are nonplanar and nonbipartite).

Inspired by a theorem of Lovász (1983), we took on the task of characterizing graphs that do not contain K_4 as a conformal minor — that is, K_4-free graphs. In a joint work with U. S. R. Murty (2016), we provided a structural characterization of planar K_4-free graphs. The problem of characterizing nonplanar K_4-free graphs is much harder, and we have evidence to believe that it has connections to the problem of recognizing Pfaffian graphs. In particular, we conjecture that every graph that is K_4-free and $K_{3,3}$-free is also Pfaffian. This conjecture is trivially true for bipartite graphs (by Little's Theorem), and also true for near-bipartite graphs due to the Fischer-Little Theorem. Using extensive computations, we have verified this conjecture, and related conjectures, for all graphs of order at most 18, and for cubic graphs of order at most 26.